On the problem whether controllability is finitely determined

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Let \( \{f_1, \ldots, f_m\} \) be an STLC family of vector fields. Is it true that:

(i) \( \exists N_1 \) such that any family of vector fields with the same Taylor polynomials of order \( N_1 \) at \( 0 \) as the Taylor polynomials of \( f_1, \ldots, f_m \) is STLC?

(ii) \( \exists N_2 \) such that \( \mathcal{A}_t \) contains a ball of radius \( t^{N_2} \) centered at \( 0 \) for any small enough \( t > 0 \)?

(iii) \( \exists N_3 \) such that \( 0 \in \text{int} \mathcal{A}_t(N_3) \), for any \( t > 0 \)?
Outline – as requested: “more tutorial”

- Background / motivation
  - Controllability
  - Nilpotent approximating systems
  - Old example: increasingly oscillatory variations

- New “counterexamples” (w/ Bianchini, 2003)
  - Lack of convexity of derivatives
  - A neat “boots-strapping” argument

- Summary
Deciding controllability

- $\dot{x} = Ax + Bu, \ x \in \mathbb{R}^n$, Kalman Rank Condition
  
a-priori known bound on $\#$ of matrix operations needed

- $\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \ x \in M^n$,
  
  want: algebraic criteria, algorithm
  
  idea (??): just calculate higher & higher derivatives of
  
  end-point map $\Phi: \mathcal{U} = L^1([0, T], [-1, 1]^m) \hookrightarrow \mathbb{R}^n$’
  
  notion derivatives $\rightarrow$ (Sussmann) GDQ

- many notions of controllability
  
  Lie algebra rank condition: only accessibility
Small-time local controllability

- importance: e.g. duality w/ (time-)optimal control

- decidability: NP-hard (poly.sys w/ rat.data) (EDS, MK)

- algorithmic algebraic conditions: much progress in 1980s
  Lobry, Krener, Hermes, Sussmann, Stefani, Bianchini, Bressan, Schättler, MK, ... little since ...

- all known general conditions are closely intertwined
  with nilpotent approximations: “leading terms”
  common confusion: STLC of approx sys $\leftrightarrow$ STLC orig sys
Nilpotent approximations versus STLC

Want: Simple (easy to integrate, analyze, control, . . .) approximating system that preserves geometric controllability/optimality properties

Also:

Proofs of general controllability/optimality conditions are fundamentally based on estimates related to nilpotent approximating systems

Caution: Nilpotent approximating systems of systems known to be STLC from general arguments will again be STLC!

But known conditions for STLC are imperfect → no known general algorithm for obtaining nilpotent approximating systems that preserves controllability indeed, existence of such algorithm is open problem basically the $Q$ whether STLC is finitely determined
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indeed, existence of such algorithm is open problem
basically the Q whether STLC is finitely determined
Nilpotent approximations – algorithmic

Algorithmic! \(\rightarrow\) ComputerAlgebraSystem

- Calculate iterated Lie brackets of \(f\) and \(g_i\), and evaluate at initial point until vectors span tangent space (enough?)
- count the number of differentiations needed \(\rightarrow\) dilation exponents (weights) \(r_i\)
- change to adapted / coordinates (Stefani 1985, Bellaïche) (triangular polynomial coord. change, w/ poly. inverse)
- truncate components of \(f\) and \(g_i\) according to weights \(r_i\)
Nilpotent approximations – a practical example

Parallel parking – no ambiguities here

\[ f_0(x) = \begin{pmatrix} 0 \\ 0 \\ x_2 \cos x_4 \\ x_2 \tan x_1 \\ x_2 \sin x_4 \end{pmatrix}, \quad f_1(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad f_2(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \]
Nilpotent approximations – algorithmic: An example

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f_0(z) = \begin{pmatrix} z_3 \cos z_4 \\ z_3 \sin z_4 \\ 0 \\ z_3 \sin z_5 \\ 0 \end{pmatrix} \quad f_1(z) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad f_2(z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
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\[
[f_1, f_0](x) = \begin{pmatrix} \cos z_4 \\ \sin z_4 \\ 0 \\ \sin z_5 \\ 0 \end{pmatrix} \quad [f_2, f_0](x) = \begin{pmatrix} 0 \\ 0 \\ z_3 \cos z_5 \\ 0 \\ 0 \end{pmatrix} \quad [f_2, f_1](x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
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[f_1, [f_2, f_0]](x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos z_5 \\ 0 \end{pmatrix} \quad [[f_1, [f_2, f_0]], [f_1, f_2]](x) = \begin{pmatrix} \sin z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}
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Dilation exponents \( r = (2, 5, 1, 3, 1) \), adapted coordinates - permutation, truncate Taylor series of components at weighted orders
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Dilation exponents \( r = (2, 5, 1, 3, 1) \), adapted coordinates - permutation, truncate Taylor series of components at weighted orders
How many more differentiations?

- Many well-studied systems: *neutralizing possible obstructions* to STLC, different weighting schemes (state-of-art: Sussmann 1987)
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- but still gap between necessary and sufficient conditions

  \[
  \text{STLC} \quad \implies \quad \text{accessible} \\
  \implies \quad \text{LARC} \\
  \implies \quad \exists N_1 < \infty \text{ s.t. brackets of length } < N_1 \text{ span} \\
  \text{(there is no (no a-priori bound) on } N_1 \text{ in terms of e.g. } n) .
  \]

Open problem:

Does there exist a function \( N_2(N_1) \) s.t. for every system \((f, g)\) with \( \dim L_{N_1}(f, g)(0) = n \), STLC can be decided by considering brackets of length at most \( N_2(N_1) \)?
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Let \{f_1, \ldots f_m\} be an STLC family of vector fields. Is it true that:

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Let \( \{f_1, \ldots, f_m\} \) be an STLC family of vector fields. Is it true that:

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Is it always possible to recognize an STLC family in a finite number of differentiations?

(ii) \( \exists N_2 \) such that \( \mathcal{A}_t \) contains a ball of radius \( t^{N_2} \) centered at 0 for any small enough \( t > 0 \)?

Is the value function of an STLC system always Hölder continuous?

(iii) \( \exists N_3 \) such that \( 0 \in \text{int} \mathcal{A}_t(N_3) \), for any \( t > 0 \)?

How complicated a control strategy does one need?

(iii) implies (i) and (ii) but is false if \( n \geq 4 \), open for \( n = 3 \).
Need for increasingly oscillatory control variations

\[ \begin{aligned}
\dot{x}_1 &= u \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= x_1^3 \\
\dot{x}_4 &= x_1^2 + x_2^7
\end{aligned} \]

(MK, Bull.AMS 1988)
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\\[ \forall N \exists T > 0 \text{ s.t. } \forall u \text{ if } \tau \mapsto \int_0^\tau u(\sigma) \, d\sigma \text{ changes sign at most } N \text{ times on } [0, T] \text{ then } x_4(T, u) \geq 0. \]
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- STLC: \( \forall T > 0, \ \mathcal{R}(T) \supseteq B(0, CT^{57}) \)
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Any standard nilpotent approximation is NOT STLC
Need for increasingly oscillatory control variations

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- STLC: \( \forall T > 0, \) 
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The system w/ higher order term \( x_2^7 \) is STLC
Need for increasingly oscillatory control variations

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\dot{x}_1 &= u \\
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\[\forall N \exists T > 0 \text{ s.t. } \forall u \text{ if } \tau \mapsto \int_{0}^{\tau} u(\sigma) \, d\sigma \text{ changes sign at most } N \text{ times on } [0, T] \text{ then } x_4(T, u) \geq 0.\]

\[\text{STLC: } \forall T > 0, \quad \mathcal{R}(T) \supseteq B(0, C T^{57})\]

Number of diff’s needed to decide STLC? STLC nilpot approx?
The next complication

Loosely speaking, control variations that are based on rescaling (time and amplitude) a compact family of control directions cannot identify all STLC systems. Need: directional derivatives along curves in space of controls that *twist out* of every finite dimensional subspace.
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Various other examples
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- various other examples

- things get worse: (Bianchini & MK 2003) higher order derivatives need not have convex images contrast w/ first order maximum principle, *separation* of approximating *cones*
Needle variations that cannot be summed

Bianchini & MK (SICON 2003)

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1^2 + (1 + u_{01}) \\
\dot{x}_4 &= x_2^2 + (1 + u_{02}) \\
\dot{x}_5 &= x_4 x_1^2 - x_1^7 \\
\dot{x}_6 &= x_3 x_2^2 - x_2^7 \\
|u_1(\cdot)| &\leq 1 \\
|u_2(\cdot)| &\leq 1 \\
|u_{01}(\cdot)| &\leq 1 \\
|u_{02}(\cdot)| &\leq 1 \\
x(0) &= 0 \\
x^*(t) &= (0, 0, t, t, 0, 0)
\end{align*}
\]
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\end{align*}

The system is STLC about the ref trajectory.

The perturbed system is not STLC about the ref trajectory.
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\dot{x}_4 &= x_2^2 + (1 + u_{02}) & |u_{02}(\cdot)| &\leq 1 \\
\dot{x}_5 &= x_4 x_1^2 - x_1^7 + x_1^{10} + x_2^{10} & x(0) &= 0 \\
\dot{x}_6 &= x_3 x_2^2 - x_2^7 + x_1^{10} + x_2^{10} & x^*(t) &= (0, 0, t, t, 0, 0)
\end{align*}
\]

The system is STLC about the ref trajectory.

The perturbed system is not STLC about the ref trajectory.
Intuitive strategy to reach \((0, 0, *, *, -, -)\)

First generate the “difficult” directions

Then generate the “easy” directions

“Hold back” along ref. trajectory

“Catch up” with the ref. reference trajectory

“Coast along” ref. trajectory
Technical lemma

\[
\begin{aligned}
\dot{x}_1 &= u & |u(\cdot)| &\leq 1 \\
\dot{x}_2 &= cx_1^2 - x_1^7 & x(0) &= 0
\end{aligned}
\]

If \( c \neq 0 \) then the system is not STLC about \( x = 0 \).

If \( T > 2 \left(\frac{8}{3}|c|\right)^{\frac{1}{5}} \) then \( 0 \in \text{int} \mathcal{R}(T) \).
Elements of the proof: Inequality I

Use invariance under the index permutation
\((x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_2, x_1, x_4, x_3, x_6, x_5)\) and
\[
\int_0^T \left( \int_0^t x_1^2(s) \, ds \right) x_2^2(t) \, dt + \int_0^T x_1^2(t) \left( \int_0^t x_2^2(s) \, ds \right) \, dt = \left( \int_0^T x_1^2(t) \, dt \right)
\]
to assume wlog that
\[
\int_0^T \left( \int_0^t x_1^2(s) \, ds \right) x_2^2(t) \, dt \geq \frac{1}{2} \left( \int_0^T x_1^2(t) \, dt \right) \left( \int_0^T x_2^2(t) \, dt \right)
\]  
(1)

Rewrite s.t. have constant weight against which \(x_2^2\) is integrated
\(\mapsto\) lemma applies)

\[
\int_0^T \left( \int_0^t x_1^2(s) \, ds \right) x_2^2(t) \, dt \geq \frac{1}{2} \int_0^T \left( \int_0^T x_1^2(s) \, ds \right) x_2^2(t) \, dt
\]
(2)
Elements of the proof: Inequality II

Suppose $x_6(T) < 0$, thus

$$0 > \int_0^T \left( x_3^2(t) x_2^2(t) - x_2^7(t) \right) dt \geq \int_0^T x_2^2(t) \left( \frac{1}{2} \int_0^T x_1^2(s) ds - x_2^5(t) \right) dt$$

and thus

$$\xi_2 \overset{\text{def}}{=} \max_{0 \leq t \leq T} x_2(t) > \left( \frac{1}{2} \int_0^T x_1^2(s) ds \right)^{\frac{1}{5}}.$$

Consider term on right hand side a cost, or energy, that is required to move $x_6(0) = 0$ to $x_6(T) < 0$.

Find lower bound for energy in terms of displacement. Crudely:

$$\int_0^T x_1^2(s) ds \geq 2 \cdot \left( \frac{1}{3} \right) \cdot \xi_1^3 \overset{\text{def}}{=} \frac{2}{3} \max_{0 \leq t \leq T} x_1(t)$$
Elements of the proof: Inequality III

Using Minkowski's inequality

\[- x_5(T) = \int_0^T (x_1^7(t) - x_4(t)x_1^2(t))dt \leq \int_0^T |x_1^7(t)|dt \leq T \cdot \xi_1^7\]

Combine the three identities

\[\xi_2 > \left(\frac{1}{2} \int_0^T x_1^2(s) ds\right)^{\frac{1}{5}} \geq \left(\frac{1}{2} \cdot \frac{2}{3} \xi_1^3\right)^{\frac{1}{5}} \geq \left(\frac{1}{3} \left(\frac{-x_5(T)}{T}\right)^{\frac{3}{7}}\right)^{\frac{1}{5}}\]

Use \(\xi_2 \leq \frac{1}{2} T\) (from \(u_2(\cdot)| \leq 1 \) and \(x_2(0) = x_2(T) = 0\))

\[-x_5(T) \leq 3^{\frac{1}{5}} \cdot \xi_2^{\frac{35}{3}} \cdot T \leq 3^{\frac{1}{5}} \cdot 2^{-\frac{35}{3}} \cdot T^{\frac{38}{3}}.\]

Consequently if \(x(T) = (0, 0, *, *, -\rho, -\rho) \in \mathcal{R}_0(T)\) with \(\rho > 0\), then \(\rho < CT^{38/3}\) (with \(C = 3^{1/5} \cdot 2^{-35/3}\)).
Summary and outlook

Something to take home:

(usual) nilpotent approximating system
versus original system:

in theory: controllability of either one does not imply
controllability of other

but in practical examples this is not an issue
Summary and outlook

- Something to take home:

  (usual) nilpotent approximating system versus original system:

  in theory: controllability of either one does not imply controllability of other

  but in practical examples this is not an issue

- Invitation: try completely different approaches to settle the fundamental open problem whether STLC is finitely determined,

  and whether one always can find a STLC preserving nilpotent approximating system