1. Consider \( A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \), \( b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \), and \( x_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \).

   a. Discuss the stability of the differential equation \( x' = Ax \). Is the system hyperbolic?
   b. Calculate the matrix exponential of \( A \).
   c. Find the solution of the initial value problem \( x' = Ax + b, x(0) = x_0 \).

   *Keep your eyes open – you may find lots of possible simplifications in b. and c.*

2. Suppose \( f : \mathbb{R}^n \mapsto \mathbb{R}^n \) is continuous.

   a. Explain why solution curves of \( x' = f(x) \) can never cross transversally.
      Give an example of a differential equation \( x' = f(x) \) some of whose solution curves cross each other tangentially.
   b. State a general theorem, with precise conditions on \( f \), that guarantees that no two solution curves of \( x' = f(x) \) cross each other.
   c. Prove the theorem stated in b.

3. Consider the second order scalar differential equation \( z'' + \frac{1-z^2}{1+z^2} z' + z = 0 \).

   a. Write the differential equation as an equivalent system of first order equations.
   b. Decide whether solutions to initial value problems are unique, and what their maximal domains are. Justify your answers.
   c. Find all equilibria and analyze their stability
      (i) using the linearization of the system about the origin, AND
      (ii) using the candidate Lyapunov function \( V(x, y) = x^2 + y^2 \) and Lasalle’s theorem.

   **BONUS.** Do you expect this system to have a nontrivial periodic orbit? Explain.

4. a. Which of the sets pictured above can be \( \omega \)-limit sets of a system \( x' = f(x) \) in the plane?
   Assume \( f : \mathbb{R}^2 \mapsto \mathbb{R}^2 \) is continuously differentiable and all positive semi-orbits are bounded.
   Justify your answer (e.g., sketch possible trajectories, or explain in detail why impossible.)
   b. For each of the possible \( \omega \)-limit sets identify all points that must be equilibria.
   c. Prove that your answer in b. for the set (i) (a closed, finite line segment) is correct.