1. a. False. If \( D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) then \( e^{t(D+U)} = \begin{pmatrix} 1 & e^t - 1 \\ 0 & e^t \end{pmatrix} \) whereas \( e^{tD}e^{tU} = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \). 

b. The only eigenvalue of \( A \) is \( -1 \), and it has algebraic multiplicity 4.

c. Define \( N = A + I \). Since rank\((N) = 2 \), the eigenspace \( E_{-1} \) has dimension 2.

It is spanned by the eigenvectors \( v_1 = (1, 0, 0, 0)^T \) and \( v_2(0, 1, 0, 0)^T \).

The only possible Jordan forms of \( A \) are \( J_1 \) and \( J_2 \) on the right.

d. Since \( N^3 = 0 \), \( e^{tN} = I_A + tN + \frac{t^2}{2} N^2 \) is in (in box) and \( e^{tA} = e^{-tI}e^{tN} = e^{-t}e^{tN} \).

2. a. For every matrix \( A \in \mathbb{R}^{n \times n} \), define \( e^A = \int e^{tA} dt \). Since \( \text{rank}(N) = 2 \), the eigenspace \( E_{-1} \) has dimension 2.

It is spanned by the eigenvectors \( v_1 = (1, 0, 0, 0)^T \) and \( v_2(0, 1, 0, 0)^T \).

The only possible Jordan forms of \( A \) are \( J_1 \) and \( J_2 \) on the right.

e. \( \text{y}(0) = (0, 5, 0, 0)^T \) is a multiple of an eigenvector, hence \( \phi(t) = (0, 5e^{-t}, 0, 0)^T \).

**BONUS.** Since \( N^2 \neq 0 \), \( J_1 \) is the Jordan form of \( A \).

3. a. For every matrix \( A \in \mathbb{R}^{n \times n} \), define \( e^A \).

b. If \( \phi' = A\phi + g \), and \( \phi(t_0) = y_0 \), then for all \( t \), \( \frac{d}{dt}(e^{tA}\phi(t)) = e^{tA}(A\phi(t) + \phi'(t)) = e^{tA}g(t) \) and hence \( \phi(t) = e^{-tA}y_0 + \int_{t_0}^{t} e^{sA}g(s) \, ds \).

4. [was recent homework – else BN section 2.8]

5. a. The function \( y \rightarrow 1 + y^2 \) is continuously differentiable and hence locally Lipshitz which guarantees existence of unique solutions on some sufficiently small open interval about \( t_0 \). On the other hand, for \( y \geq 1, 1 + y^2 \geq 1 + y^2 \) and hence solutions escape to infinity even faster than those of \( y' = 1 + y^2 \), i.e. before \( t = \frac{\pi}{2} \).

b. The right hand side of the DE \( y' = -2\text{sgn}(y - \sin t) \) is not continuous along the curve \( y = \sin t \), and it is smooth everywhere else, in particular, locally Lipshitz continuous at all point not on this curve. Thus there is a unique solution starting at \( (0, 0) \) which decreases until it hits the curve \( y = \sin t \), at which point none of our theorems apply anymore.

c. The right hand side of the DE \( y' = 1 + y^{2/3} \) is continuous in \((t, y)\) and locally Lipshitz at all points \((t, y)\) where \( y \neq 0 \). Thus there exists a unique solution starting at \( y(0) = -1 \) until this solution hits the curve \( y = 0 \) at which point our theorems no longer apply. (It turns out that unique solutions exist globally on \( \mathbb{R} \).)

6. a. [BN p.31, theorem 1.4] or [W p.178 lemma 4.4].

b. Formally introduce \( y_1 = x, y_2 = y' \). The initial value problem \( y_1' = y_2, y_2' = -\frac{1}{2}y_1 \). The initial value problem \( y_1(0) = 1, y_2(0) = 0 \).

c. For all \( t \) sufficiently close to \( t_0 = 1 \), \( \phi_0(t) = (t - 1, 1 - \log t)^T \), \( \phi_0(t) = (t - 1, 1 - \log t)^T \), \( \phi_0(t) = (t - 1, 1 - \log t)^T \), \( \phi_0(t) = (2t - t \log t - 2, 1 - \log t + \frac{1}{2}(\log t)^2)^T \), \( \phi_0(t) = (3(2 - t \log t + \frac{1}{2}(\log t)^2 - 3 - 1 \log t + \frac{1}{2}(\log t)^2)^T \).