27/5. Supp. \( p \in \mathbb{R}^n \) is such that there exists \( T > 0 \) s.t. \( \psi(T, p) = p \). ("periodic orbit").

Let \( C^+_p = \{ \psi(t, p) : t \geq 0 \} \) be the positive semi-orbit thru \( p \) and \( \omega(p) \) its limit set.

To show \( C^+_p \subseteq \omega(p) \), note that if \( q \in C^+_p \) then \( \exists t_0 \) s.t. \( \psi(t_0, p) = q \). Consequently for every \( k \in \mathbb{Z}^+ \)

\[
\psi(t_0 + kT, p) = \psi(t_0, \psi(kT, p)) = \psi(t_0, q) = q
\]

and hence \( q \in \omega(p) \).

Conversely, to show \( \omega(p) \subseteq C^+_p \), suppose \( z \notin C^+_p \). Since \( t \mapsto \psi(t, p) \) is continuous and \([0, T]\) is compact, so is

\[
C^+_p = \{ \psi(t, p) : 0 \leq t \leq T \}.
\]

Since it is closed, \( \exists \varepsilon > 0 \) s.t.

\[
B(z, \varepsilon) \cap C^+_p = \emptyset,
\]

hence \( z \notin \omega(p) \).

Alternatively, consider \( g : [0, T] \to \mathbb{R}^n \) defined by \( g(t) = \|z - \psi(t, p)\| \).
2.7/6 \[ \text{True in } \mathbb{R}^2 \text{ only??} \]
Supp. \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) is \( C^1 \) and there are no periodic solutions of \( x' = f(x) \).
Supp. \( p \in \mathbb{R}^2 \) and \( \{ \Phi(t, p) : t \geq 0 \} \) is bounded and hence \( \omega(p) \neq \emptyset \).

If \( \omega(p) \) contains no equilibria, then by P.B. it must be a periodic orbit.
Since no such exist, \( \omega(p) \) must contain an equilibrium of \( x' = f(x) \).
2.7/7 Supp. $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ and $p \in \mathbb{R}^n$ is such that the solution $t \to \phi(t, p) ; t \geq 0$ is bounded and $\omega(p)$ contains an asymptotically stable equilibrium point $q$.

Supp. $z \in \mathbb{R}^n \setminus \{q\}.$

Let $\varepsilon = \|z - q\|. \text{ Clearly } \varepsilon > 0.$

Hence $\exists \delta \text{ s.t. if } \|y_0 - q\| < \delta \text{ then }$

$\forall t \geq 0, \|\phi(t, y_0) - q\| < \frac{\varepsilon}{2}.$

Since $q \in \omega(p), \exists T, \text{ s.t. } \|\phi(T, p) - q\| < \delta.$

Hence for all $t > T$, $\|\phi(t, p) - q\| < \frac{\varepsilon}{2}$ and thus

for all $t > T$

$\|\phi(t, p) - z\| > \frac{\|z\|}{2}$

and hence $z \notin \omega(p)$. 
2.7/8 \[ \begin{align*}
    x' &= y \\
    y' &= 2(1 - xy)
\end{align*} \]

Has no equilibria:
\[ \begin{align*}
    x' = 0 &\quad \Rightarrow \quad y = 0 \\
    y' = 0 &\quad \Rightarrow \quad xy = 1
\end{align*} \] no solution.

But by theorem 7.3, every periodic trajectory of a planar system contains a critical point in its interior [sic].

Since are no critical points, there can be no periodic solutions either.