Remarks on asymptotic stability and $KL$ functions

Technically, notions of attractivity and stability do not require uniqueness of solutions of initial value problems. But as the following ill-fated attempts to give precise definitions show, this generality is not really worth the effort. If really needed, one always can define it just in time.

Definition: Suppose $E \subseteq \mathbb{R}^n$ is open, $0 \in E$ and $F: E \mapsto \mathbb{R}^n$ is continuous. An equilibrium point $x_0 \in E$, i.e. $F(x_0) = 0$ is called

- (locally) attractive if there exists an open neighborhood $U \subseteq E$ of $x_0$ and for every $\varepsilon > 0$ and every $x \in U$ all solutions of $\dot{x} = F(x)$, $x(0) = x_0$ exist for all positive times and stay in $E$, and there exists $T > 0$ such that for all $t > T$, and for all $y_0 \in U$ and for every solution $\phi: [0, \infty) \mapsto \mathbb{R}^n$ of $\dot{x} = F(x)$ that satisfies $\phi(0) = y_0$, $\|\phi(t)\| < \varepsilon$.

- (Lyapunov) stable if for every open neighborhood $U \subseteq \mathbb{R}^n$ of 0 there exists an neighborhood $V \subseteq \mathbb{R}^n$ of 0 such that for all $t \geq 0$, and every $y_0 \in V$ and every solution $\phi: [0, \infty) \mapsto \mathbb{R}^n$ of $\dot{x} = F(x)$ that satisfies $\phi(0) = y_0$, $\phi(t) \in U$.

- (locally) asymptotically stable if it is (locally) attractive and Lyapunov stable.

Example. The origin is globally attractive for the system defined below, but it is not Lyapunov stable (Hahn, p. 191 – a quite lengthy argument!)

$$
\dot{x} = \frac{x^2(y-x) + y^5}{(x^2 + y^2)(1 + (x^2 + y^2)^2)}, \quad \dot{y} = \frac{y^2(y-2x)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)}.
$$

Exercise. Calculate the partial derivatives of the vector field defined by this system at the origin, and discuss its differentiability. Sketch the vector field, together with the null-clines, and identify invariant subsets.

Definition (Hahn):

- Denote by $\mathcal{K}$ the class of continuous functions $f: [0, \infty) \mapsto [0, \infty)$ that are strictly increasing and satisfy $f(0) = 0$.

- Denote by $\mathcal{K}_\infty$ the class of unbounded functions in $\mathcal{K}$.

- Denote by $KL$ the class of functions $f: [0, \infty)^2 \mapsto [0, \infty)$ that are of class $\mathcal{K}$ in the first argument, and decrease to zero in the second argument.

Exercise. Suppose that $F: \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous and $F(0) = 0$. Show that $x_0 = 0$ is an asymptotically stable equilibrium point of $\dot{x} = F(x)$ if and only if there exists a function $\beta$ of class $KL$ such that for all $x \in \mathbb{R}^n$ and all $t \geq 0$ and for every solution $\phi: [0, \infty) \mapsto \mathbb{R}^n$ of $\dot{x} = F(x)$ that satisfies $\phi(0) = x$

$$
\|\phi(t)\| \leq \beta(\|x\|, t).
$$