Poincaré–Bendixson Theorem

Suppose \( D \subseteq \mathbb{R}^2 \) open, \( f : D \to \mathbb{R}^2 \) loc. Lipschitz.

If \( \Omega \subseteq D \) is a nonempty, closed, bounded positive limit set of \( y' = f(y) \) that contains no equilibrium point then \( \Omega \) is a periodic orbit.

[e.g. Hirsch–Smale–Devaney p. 225, or Khalil p. 290]

Suppose \( S \subseteq D \) is closed and bounded and contains no equilibrium point of \( y' = f(y) \). If \( p \in S \) is such that \( \forall t \geq 0, \varphi(t, p) \in S \), then either \( \{ \varphi(t, p) : t \geq 0 \} \) is a limit cycle or \( \varphi(\cdot, p) \) converges to a limit cycle.

[e.g. wikipedia, Wattman p. 144]
Hence, for all \( x \) in the ball \( B_\delta(y_0) \), the trajectory starting at \( x \) crosses \( L \) at \( \tau(x) \in (-\epsilon, \epsilon) \). If the trajectory \( \phi(t, x) \) is bounded, then

\[
||\phi(\tau(x), x) - y_0|| = ||\phi(\tau(x), x) - x + x - y_0|| \\
\leq ||\phi(\tau(x), x) - \phi(0, x)|| + ||x - y_0|| \\
\leq k|\tau(x)| + \delta < k\epsilon + \delta
\]

where \( k \) is a bound on \( f(\phi(t, x)) \). Since, without loss of generality, we can always choose \( \delta < \epsilon \), the right-hand side of the last inequality can be made arbitrarily small by choosing \( \epsilon \) small enough. \( \Box \)

**Lemma A.5** If a trajectory \( \gamma \), followed as \( t \) increases, crosses a transversal at three consecutive points \( y_1, y_2, \) and \( y_3 \), then \( y_2 \) lies between \( y_1 \) and \( y_3 \) on \( L \). \( \triangle \)

**Proof:** Consider the two consecutive points \( y_1 \) and \( y_2 \). Let \( C \) be a simple closed (Jordan) curve made up of the part of the trajectory between \( y_1 \) and \( y_2 \) (the arc \( MNP \) in Figure A.1) and the part of the transversal \( L \) between the same points (the segment \( PQM \) in Figure A.1). Let \( D \) be the closed bounded region enclosed by \( C \). We assume that the trajectory of \( y_2 \) enters \( D \); if it leaves, the argument is similar. By uniqueness of solutions, no trajectory can cross the arc \( MNP \). Since \( L \) is a transversal, trajectories can cross \( L \) in only one direction. Hence, trajectories will enter \( D \) along the segment \( PQM \) of \( L \). This means the set \( D \) is positively invariant. The trajectory of \( y_2 \) must remain in the interior of \( D \) which, for the case sketched in Figure A.1, implies that any further intersections with \( L \) must take place at a point below \( y_2 \). \( \Box \)

The next lemma is concerned with the intersection of limit sets and transversals.

**Lemma A.6** Let \( y \) be a positive limit point of a bounded positive semi-orbit \( \gamma^+ \). Then, the trajectory of \( y \) cannot cross a transversal at more than one point. \( \triangle \)
A.10 Proof of Theorem 7.1

We prove the Poincaré-Bendixson theorem only for positive limit sets. The proof for negative limit sets is similar. We start by introducing transversals with respect to a vector field \( f \). Consider the second-order equation

\[
\dot{x} = f(x)
\]  
(A.20)

where \( f : D \to \mathbb{R}^2 \) is locally Lipschitz over a domain \( D \subset \mathbb{R}^2 \). A transversal with respect to \( f \) is a closed line segment \( L \in D \) such that no equilibrium points of (A.20) lie on \( L \) and at every point \( x \in L \), the vector field \( f(x) \) is not parallel to the direction of \( L \). If \( L \) is a segment of a line whose equation is

\[
g(x) = a^T x - c = 0
\]

then \( L \) is a transversal of \( f \) if

\[
a^T f(x) \neq 0, \quad \forall x \in L
\]

If a trajectory of (A.20) meets a transversal \( L \), it must cross \( L \). Moreover, all such crossings of \( L \) are in the same direction. In the next two lemmas, we state the properties of the transversals which will be used in the proof.

Lemma A.4 If \( y_0 \) is an interior point of a transversal \( L \), then for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that any trajectory passing through the ball \( B_\delta(y_0) = \{ x \in \mathbb{R}^2 \mid \| x - y_0 \| < \delta \} \) at \( t = 0 \) must cross \( L \) at some time \( t \in (-\varepsilon, \varepsilon) \). Moreover, if the trajectory is bounded, then by choosing \( \varepsilon \) small enough the point where the trajectory crosses \( L \) can be made arbitrarily close to \( y_0 \).

\[\Delta\]

Proof: Let \( \phi(t, x) \) denote the solution of (A.20) that starts at \( \phi(0, x) = x \), and define

\[
G(t, x) = g(\phi(t, x)) = a^T \phi(t, x) - c
\]

The trajectory of \( \phi(t, x) \) crosses \( L \) if \( G(t_1, x) = 0 \) for some time \( t_1 \). For the function \( G(t, x) \) we have \( G(0, y_0) = 0 \) since \( y_0 \in L \), and

\[
\frac{\delta G}{\delta t}(0, y_0) = a^T f(\phi(t, y_0)) \bigg|_{t=0} = a^T f(y_0) \neq 0
\]

since \( L \) is a transversal. By the implicit function theorem, there is a continuously differentiable function \( \tau(x) : U \to \mathbb{R} \) defined on a neighborhood \( U \) of \( y_0 \) such that \( \tau(y_0) = 0 \) and \( G(\tau(x), x) = 0 \). By continuity of the map \( \tau(x) \), given any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
\| x - y_0 \| < \delta \Rightarrow | \tau(x) | < \varepsilon
\]
Proof: Suppose that $y_1$ and $y_2$ are two distinct points on the trajectory of $y$ and $L$ is a transversal containing $y_1$ and $y_2$. Suppose $y \in \gamma^+(x_0)$ and let $x(t)$ denote the solution starting at $x_0$. Then, $y_k \in L^+(x_0)$ for $k = 1, 2$ because Lemma 3.1 proves the positive limit set $L^+$ is invariant. Let $J_k \subset L$ be an interval that contains $y_k$ in its interior. Assume that $J_1$ and $J_2$ are disjoint (Figure A.2). By Lemma A.4, given any $\epsilon > 0$ there is $\delta > 0$ such that any trajectory that comes within a distance $\delta$ from $y_1$ will cross $L$. Since $y_1$ is a positive limit point of $x(t)$, there is a sequence $t_n$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) \to y_1$ as $n \to \infty$. The solution $x(t)$ enters the ball $B_{\delta}(y_1)$ infinitely often; hence, it crosses $L$ infinitely often. Let the sequence of crossing points be $x_n$, ordered as $t$ increases. Since the trajectory is bounded, we can choose $\epsilon$ small enough to ensure that $x_n$ is arbitrarily close to $y_1$. In particular, given any $\delta > 0$, there is $N > 0$ such that for all $n \geq N$, $\|x_n - y_1\| < \delta$. This shows that $x_n \to y_1$ as $n \to \infty$ and, for sufficiently large $n$, the sequence of crossing points $x_n$ will lie within the interval $J_1$. By the same argument, the solution crosses $J_2$ infinitely often. Thus, there is a sequence of crossing points $a_1, b_1, a_2, b_2, \ldots$, taken as $t$ increases, with $a_i \in J_1$ and $b_i \in J_2$. By Lemma A.5, the crossing points must be ordered on $L$ in the same order $a_1, b_1, a_2, b_2, \ldots$. However, this is impossible since $J_1$ and $J_2$ are disjoint. Therefore, the two crossing points $y_1$ and $y_2$ must be the same point. □

The last lemma states a property of bounded positive limit sets.

**Lemma A.7** Let $L^+$ be the positive limit set of a bounded trajectory. If $L^+$ contains a periodic orbit $\gamma$, then $L^+ = \gamma$. △

Proof: Let $L^+ = L^+(x)$ for some point $x$. It is enough to show that

$$\lim_{t \to \infty} \text{dist}(\phi(t, x), \gamma) = 0$$

where $\text{dist}(\phi(t, x), \gamma)$ is the distance from the trajectory of $x$ to $\gamma$. Let $L$ be a transversal at $z \in \gamma$, so small that $L \cap \gamma = z$. By repeating the argument in the
proof of the previous lemma, we know that there is a sequence \( t_n \) with \( t_n \to \infty \) as \( n \to \infty \) such that

\[
x_n = \phi(t_n, x) \in L
\]

\[
x_n \to z \quad \text{as} \quad n \to \infty
\]

\[
\phi(t, x) \notin L \quad \text{for} \quad t_{n-1} < t < t_n, \quad n = 1, 2, \ldots
\]

By Lemma A.5, \( x_n \to z \) monotonically in \( L \). Since \( \gamma \) is a periodic orbit, \( \phi(\lambda, z) = z \) for some \( \lambda > 0 \). For \( n \) sufficiently large, \( \phi(\lambda, x_n) \) will be within the ball \( B_\delta(z) \) (as defined in Lemma A.4); hence, \( \phi(t + \lambda, x_n) \in L \) for some \( t \in (-\epsilon, \epsilon) \). Thus,

\[
|t_{n+1} - t_n| < \lambda + \epsilon
\]

which gives an upper bound for the set of positive numbers \( t_{n+1} - t_n \). By continuous dependence of the solution on initial states, given any \( \beta > 0 \) there is \( \delta > 0 \) such that if \( ||x_n - z|| < \delta \) and \( |t| < \lambda + \epsilon \), then

\[
||\phi(t, x_n) - \phi(t, z)|| < \beta
\]

Choose \( n_0 \) large enough that \( ||x_n - z|| < \delta \) for all \( n \geq n_0 \). Then, the last inequality holds for \( n \geq n_0 \). Now, for all \( n \geq n_0 \) and \( t \in [t_n, t_{n+1}] \) we have

\[
\text{dist}(\phi(t, x), \gamma) \leq ||\phi(t, x) - \phi(t - t_n, z)|| = ||\phi(t - t_n, x_n) - \phi(t - t_n, z)|| < \beta
\]

since \( |t - t_n| < \lambda + \epsilon \). \( \Box \)

We are now ready to complete the proof of the Poincaré-Bendixson theorem. Since \( \gamma^+ \) is a bounded positive semiorbit, by Lemma 3.1, its positive limit set \( L^+ \) is a nonempty, compact, invariant set. Let \( y \in L^+ \) and \( z \in L^+(y) \subset L^+ \). Define a transversal \( L \) at \( z \); notice that \( z \) is not an equilibrium point because \( L^+ \) is free of equilibrium points. By Lemma A.6, the trajectory of \( y \) cannot cross \( L \) at more than one point. On the other hand, there is a sequence \( t_n \) with \( t_n \to \infty \) as \( n \to \infty \) such that \( \phi(t_n, y) \to z \). Hence, the trajectory of \( y \) crosses \( L \) infinitely often. Since there can be only one crossing point, the sequence of crossing points must be a constant sequence. Therefore, we can find \( r, s \in R \) such that \( r > s \) and \( \phi(r, y) = \phi(s, y) \). Since \( L^+ \) contains no equilibrium points, the trajectory of \( y \) is a periodic orbit. It follows from Lemma A.7 that \( L^+ \) is a periodic orbit, since it contains a periodic orbit.