Theorem. Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ is open, and $f: \Omega \to \mathbb{R}^n$ is continuous. For every continuously differentiable function $\varphi: I_0 \to \mathbb{R}^n$ defined on a nonempty open interval $I_0$ whose graph lies inside $\Omega$ and which satisfies for all $t \in I_0$, $\varphi'(t) = f(t, \varphi(t))$, there exists an extension $\overline{\varphi}: J \to \mathbb{R}^n$ of $\varphi$ to a maximal interval $J \supseteq I_0$ whose graph lies inside $\Omega$ and which satisfies for all $t \in J$, $\overline{\varphi}'(t) = f(t, \overline{\varphi}(t))$.

Proof. Suppose $\Omega, f, I_0, \varphi$ have the stated properties and relationships. Consider the collection $\mathcal{F}$ of all pairs $(I, \psi)$ that have the following properties: $I \subseteq \mathbb{R}$ is an open interval that contains $I_0$, $\psi: I \to \mathbb{R}^n$ is a continuously differentiable extension of $\varphi$, the graph of $\psi$ lies inside $\Omega$, and for all $t \in I$, $\psi'(t) = f(t, \psi(t))$.

Endow the set $\mathcal{F}$ with a partial order $\preceq$ by setting $(I_1, \psi_1) \preceq (I_2, \psi_2)$ if and only if $I_1 \subseteq I_2$ and for all $t \in I_1$, $\psi_1(t) = \psi_2(t)$. Suppose $\mathcal{C} \subseteq \mathcal{F}$ is a strictly ordered subset. Note that if $(I_1, \psi_1), (I_2, \psi_2) \in \mathcal{C}$ and $I_1 = I_2$ then necessarily $\psi_1 = \psi_2$. Define $I = \pi_1(\mathcal{C}) = \{I \subseteq \mathbb{R}: \exists \psi_I: I \to \mathbb{R}^n$ such that $(I, \psi_I) \in \mathcal{C}\}$.

By the aforementioned remark, for every $I \in I$ the function $\psi_I$ such that $(I, \psi_I) \in \mathcal{C}$ is uniquely determined.

Define $J = \bigcup_{I \in I} I$. Then for every $t \in J$, there exists an $I \in I$ such that $t \in I$.

Moreover, if $(I_1, \psi_1), (I_2, \psi_2) \in \mathcal{C}$ such that $t \in I_1 \cap I_2$, since $\mathcal{C}$ is linearly ordered, $\psi_1(t) = \psi_2(t)$ and hence one may define a function $\overline{\psi}: J_0 \to \mathbb{R}^n$ by setting $\overline{\psi}(t) = \psi_I(t)$ for any $I \in I$ such that $t \in I$.

It is immediate to see that the pair $(J_0, \overline{\psi})$ lies in $\mathcal{F}$ and is an upper bound for $\mathcal{Z}$. Therefore, by the Hausdorff Maximal Principle the collection $\mathcal{F}$ has a maximal element $(J, \overline{\varphi})$ which has the desired properties.

A fallacious shortcut that tries to avoid the use of the Hausdorff Maximal Principle starts as above, but defines $I = \pi_1(\mathcal{F})$ and again sets $J = \bigcup_{I \in I} I$. However, there is no reason why there should exist any $\psi: J \to \mathbb{R}^n$ such that $(J, \psi) \in \mathcal{F}$. Note that such approach would actually yield a largest (as opposed to a maximal) extension. To see what may go wrong, consider the explicit counterexample worked below.

Consider the open set $\Omega = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (0, 1)) \subseteq \mathbb{R}^2$ and $f: \Omega \to \mathbb{R}$ defined by $f(t, y) = 3y^{2/3}$.

In classical language consider the initial value problem

$$
\Sigma: \quad y' = f(t, y), \quad y(-2) = 0 \quad \text{for } (t, y) \in \Omega.
$$

For all $a, b \in \{-\infty\} \cup \mathbb{R}$ with $a < b$ define $\psi_{a,b}: \mathbb{R} \to \mathbb{R}$ by

$$
\psi_{a,b} = \begin{cases} 
(t - a)^3 & \text{if } t \leq a \\
0 & \text{if } a \leq t \leq b \\
(t - b)^3 & \text{if } b \leq t 
\end{cases}
$$

Each function $\psi_{a,b}$ is twice continuously differentiable and satisfies for all $t \in \mathbb{R}$, $\psi_{a,b}'(t) = 3\psi_{a,b}^{2/3}$. For every $\beta \in \mathbb{R}$ let $S_\beta$ be the set of all restrictions to open intervals of the functions $\psi_{a,b}$ with $a < -2$ whose graphs lie inside $\Omega$, i.e.,

$$
S_\beta = \{\psi_{a,b}|_{(a, \beta)}: -1 \in (a, \beta) \subseteq (a, b), \forall t \in (a, \beta), (t, \psi_{a,b}(t)) \in \Omega\}, \text{ and } a, b \in \{-\infty\} \cup \mathbb{R}
$$

If $\varphi$ is any solution of the initial value problem $\Sigma$ whose domain includes 0, then $\varphi(0) > 0$ and hence its domain must be contained in $(-\infty, 1 - (\varphi(0))^{1/3})$. Consequently, for every $\beta \geq 1$, $S_\beta = \emptyset$. On the other hand, for every $\beta \in (-1, 1)$, $S_\beta \neq \emptyset$ since for each $\beta < 1$, $S_\beta$ contains the restriction $\varphi_\beta \in S_\beta$ of $\psi_{-\infty, \beta-1}$ to the interval $(-\infty, \beta)$. Summarizing, for every $\beta \in [0, 1)$ there exists a solution of the initial value problem $\Sigma$ that is defined on the interval $I_\beta = (-\infty, \beta)$. However, there does not exist a solution of this initial value problem $\Sigma$ that is defined on the interval $J = \bigcup_{\beta \in [0, 1)} I_\beta = (-\infty, 1)$.

In particular, each solution can be extended to a maximal domain, but the zero solution $\psi: (-\infty, -1) \to 0$ does not have an extension to a largest domain. It is easy to modify the example so that instead of artificially cutting off the domain $\Omega$ along the line $y = 1$, every nontrivial solution escapes to infinity in finite time.

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