Thm. 5.1. \[ \text{Supp. } \Omega \subseteq \mathbb{R}^n \text{ open, } 0 \in \Omega, \text{ and } \]
\[ f : \Omega \to \mathbb{R}^n \text{ loc. Lipschitz and } \]
\[ V : \Omega \to \mathbb{R}, C^1 \text{ and pos. def. and } \]
\[ \dot{V} = \sum_{i=1}^{n} (D_i V) \cdot f_i \leq 0. \]
Then \[ x = f(x) \] is stable at 0.

Thm. 5.2

If in addition \( \forall x \in \Omega, \)
\[ \text{if } x \neq 0 \text{ then } \dot{V}(x) < 0, \]
then \[ x = f(x) \] is locally asymptotically stable at 0.

Proofs [BN p.205].

1. Existence
2. Stability
3. Asy-stability
Proofs of Lyapunov theorems.

[use p. 10, 24 "region" = open set]

\[ \exists r > 0 \text{ s.t. } B_r(0) \subseteq \Omega \]

\[ \exists r > 0 \text{ s.t. } \forall y \in B_r(0) \setminus \{0\}, \quad V(y) > 0 \]

\[ V(y) \leq 0 \]

\( (B_r(0) \subseteq \Omega) \)

For \( y_0 \in B_r(0) \) consider \( \Phi(t, y_0) \)

- have local existence \( \checkmark \)

- certainly have existence as long as \( \forall t < t_1 \), \( \| \Phi(t, y_0) \| < r \)

Let \( t_1 \in (0, \infty] \) be maximal

\( \forall t < t_1 \), \( \| \Phi(t, y_0) \| < r \)

Use fundamental theorem of calc.

\[ \forall t < t_1 \]

\[ V(\Phi(t, y_0)) = V(y_0) + \int_0^t V(\Phi(s, y_0)) \, ds \]

\( \leq V(y_0) \).

\( \rightarrow \) but want bound on \( \| \Phi(t, y_0) \| \)

not just on \( V(\Phi(t, y_0)) \)
Let $\epsilon > 0$ be given, $\epsilon < r$.

$A_{\epsilon, r} = \{ y \in \mathbb{R}^n : \epsilon \leq \| y \| \leq r \}$

closed & bounded

$\Rightarrow$ compact.

Use that $V$ is continuous and $0 \in A_{\epsilon, r}$

$\Rightarrow \exists \mu = \min_{y \in A_{\epsilon, r}} V(y) > 0$.

Since $V$ is cont. at 0, $\exists \delta > 0$ s.t.

$\forall y_0$, if $\| y_0 - 0 \| < \delta$ then $| V(y_0) - V(0) | < \mu$

i.e. if $\| y_0 \| < \delta$ then $\| V(y) \| < \mu$

$\therefore \forall y_0 \text{ w/ } \| y_0 \| < \delta, \forall t < t^*$

$V(\Phi(t, y_0)) \leq V(y_0) < \mu$

and consequently

$\| \Phi(t, y_0) \| \in S_{\mu} \subseteq B_{\epsilon}(0)$

proving stability and global existence.
Now assume in addition $V < 0$.  

\[ \text{on } \Omega \setminus \{0\} \]

Have: \( \forall y_0 \in B_\delta (0), \; \psi (t, y_0) \) exists for all \( t \geq 0 \) and \( t \to V (\psi (t, y_0)) \) is bounded below by zero and monotonically decreasing, and hence has a limit.

Need to show this limit is zero.

Indirect:

Supp. \( \exists y_0 \in B_\delta (0), \; \exists \eta > 0 \) s.t. \( \forall t \geq 0 \)

\[ V (\psi (t, y_0)) > \eta > 0. \]

By continuity of \( V \) at 0 \( \exists \delta' > 0 \) s.t.

\( \forall y, \; \text{if } \| y - 0 \| < \delta', \text{ then } \| V(y) - V(0) \| < \eta. \)

i.e. \( \text{if } \| y \| < \delta', \text{ then } \| V(y) \| < \eta. \)

Consider the closed annulus should have \( r' < r \)

\[ A_{\delta, r} = \{ y : \delta \leq \| y \| \leq r \} \subseteq \Omega \]

Using continuity of \( V \), and that \( 0 \notin \Omega_{\delta, r} \)

\( \exists \mu = \min_{y \in \partial \Omega_{\delta, r}} - V(y) > 0. \)
hence \( \forall t \geq 0, \; V(\Psi(t, y_0)) \leq -\mu < 0 \)
and hence
\[
V(\Psi(t, y_0)) \leq V(y_0) - \mu t.
\]
\( \downarrow \)
to \( \forall t, \; V(\Psi(t, y_0)) > \eta. \)

\[ \rightarrow \text{Assumption } \exists \eta \text{ s.t. is false,} \]
i.e. \( \forall y_0 \; \exists T \text{ s.t. } \forall t \geq T \)
\[ V(\Psi(t, y_0)) < \eta. \]