7.4. Superposition principle for (generally nonautonomous) linear systems.

Suppose $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is continuous and $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$

$$\Phi(0, x) = 0 \quad \text{and} \quad \frac{d}{dt} \Phi(t, x) = A(t) \Phi(t, x),$$

i.e. for each $x \in \mathbb{R}^n$, $t \mapsto \Phi(t, x)$ is a solution of the IVP $\dot{z} = Az, \ z(0) = x$.

We use the uniqueness theorem of ch. 7 to conclude that for all $c, t \in \mathbb{R}$, all $x_0, y_0 \in \mathbb{R}^n$

$$\Phi(t, x_0 + cy_0) = \Phi(t, x_0) + c \cdot \Phi(t, y_0).$$

Indeed it suffices to verify that both

$t \mapsto \Phi(t, x_0 + cy_0)$, and

$t \mapsto \Phi(t, x_0) + c \Phi(t, y_0)$ are solutions of the same IVP $\dot{z} = Az$ and $z(0) = x_0 + cy_0$. 
