Gronwall inequality

Suppose \( x, C, K \geq 0 \) and \( f : [0, x] \rightarrow [0, \infty) \) is continuous and satisfies for all \( t \in [0, x] \)

\[
f(t) \leq C + \int_{0}^{t} K \cdot f(s) \, ds.
\]

Then for all \( t \in [0, x] \)

\[
f(t) \leq C \cdot e^{Kt}.
\]

\( \blacksquare \): First consider \( C > 0 \).

Define \( F : [0, x] \rightarrow [0, \infty) \) by

\[
F(t) = C + \int_{0}^{t} K \cdot f(s) \, ds.
\]

Note: \( F(0) = C \) and \( F \geq f \).

Differentiate to obtain

\[
F' = K \cdot f \leq K \cdot F.
\]

and hence for all \( t \)

\[
0 \geq (F(t') - KF(t)) e^{-Kt} = \frac{d}{dt} (F(t) e^{-Kt})
\]

\( \therefore t \rightarrow F(t) e^{-Kt} \) is decreasing, and

\( \forall t \), \( f(t) \leq F(t) \leq F(0) e^{Kt} = C e^{Kt} \).