When trying to adapt Lyapunov methods for stability to an instability theorem, a first thought might be to keep $V$ positive definite and argue with indefinite $\dot{V}$. But unlike in stability theorems, one does NOT need to consider all trajectories. Hence it suffices to identify a suitable region in which both $V$ and $\dot{V}$ have the same sign. The following statement and arguments use the delicate intersection of a closed set $F$ with the open set $V^{-1}(0, \infty)$.

**Theorem** [adapted from Chetaev 1934]. Suppose $E \subseteq \mathbb{R}^n$ is open, $0 \in E$, $f: E \mapsto \mathbb{R}^n$ and $V: E \mapsto \mathbb{R}$ are continuously differentiable, $f(0) = 0$, $V(0) = 0$. Suppose $U$ an open neighborhood of 0 with compact closure $F = \overline{U} \subseteq E$ such that the restriction of $\dot{V} = (DV)f$ to $F \cap V^{-1}(0, \infty)$ is strictly positive. If for every open neighborhood $W \subseteq \mathbb{R}^n$ of 0 the set $W \cap V^{-1}(0, \infty)$ is nonempty, then the origin is an unstable equilibrium of $\dot{x} = f(x)$.

**Proof.**

1. Assume above hypotheses.
2. Let $W$ be an open neighborhood of $0 \in \mathbb{R}^n$.
3. There exists $z \in W \cap V^{-1}(0, \infty)$. Define $a = V(z) > 0$.
4. The set $K = F \cap V^{-1}[a, \infty)$ is compact, and it is nonempty since $z \in K$.
5. Using the continuity of $V$ and $\dot{V}$ there exist $m = \min_{x \in K} \dot{V}(x)$ and $M = \max_{x \in K} V(x)$.
6. Since $K = F \cap V^{-1}[a, \infty) \subseteq F \cap V^{-1}(0, \infty)$ it follows that $m > 0$.
7. Let $I \subseteq [0, \infty)$ be the maximal interval of existence for $\phi: I \mapsto E$ such that $\phi(0) = z$, and $\phi' = f \circ \phi$.
8. Define the set $T = \{t \in I: \text{for all } s \in [0, t], \phi(s) \in K\}$.
9. For all $t \in T$, $\frac{d}{dt}(V \circ \phi)(t) = (\dot{V} \circ \phi)(t) \geq m$.
10. Hence for all $t \in T$ $V(\phi(t)) = V(\phi(0)) + \int_0^t \dot{V}(\phi(s)) \, ds \geq a + mt$.
11. Since $V$ is bounded above by $M$, $T$ is also bounded above. Let $t_0$ the the least upper bound of $T$.
12. Since $K \subseteq E$, $I \supseteq [0, t_0]$ and $\phi(t_0)$ is well-defined. Moreover $t_0$ lies in the interior of $I$.
13. Since for all $t \in [0, t_0)$, $\phi(t) \in K$, and for every $\delta > 0$ there exist $t \in I$, $t_0 < t < t_0 + \delta$ such that $\phi(t) \notin K$ it follows that $\phi(t_0)$ lies on the boundary of the set $K$.
14. The boundary of $K$ is contained in the union $\partial K \subseteq V^{-1}(a) \cup (F \setminus U)$.
15. Since $V(\phi(t_0)) \geq a + mt_0 > a$ it follows that $\phi(t_0) \notin V^{-1}(a)$.
16. Therefore $\phi(t_0) \in (F \setminus U)$, and, in particular, $\phi(t_0) \notin U$.
17. Since $W$ was arbitrary, this shows that the origin is an unstable equilibrium point of $f$. ■