The big story. Throughout mathematics, a central question is “When are two objects basically the same?” Immediately following is “Among all objects that are basically the same, how to choose a natural representative?” At the elementary level, we say two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) denote the same quantity if \( ad = bc \). At our level this question concerns pairs of systems of differential equations. Abstractly, we consider equivalence relations, and synonymously, partitions of the set of all differential equations, which may be characterized by invariants. A very crude level of classification distinguishes systems by their order, or by the dimension of the underlying space. One of the most fruitful lines of inquiry focuses on actions of (semi-)groups on sets. A nontrivial elementary example is the action of the multiplicative semi-group of non-zero integers on pairs on pairs (i.e. with determinant equal to 1).

For nonlinear systems, one naturally goes beyond linear coordinate changes. One may consider global actions of (semi-)groups on sets. A nontrivial elementary example is the action of the multiplicative semi-group of non-zero integers on pairs on pairs (i.e. with determinant equal to 1). Alternatively one might consider analogous \( H \) that map orbits to orbits, i.e. not necessarily preserve the \( \tau \)-parameterization of the solution curves, i.e. one allows for suitable strictly monotone maps \( \tau_x : (a_x, b_x) \mapsto \mathbb{R} \) such that

\[
\forall x \in U, \forall t \in (a_x, b_x), \quad H(\tau(t, x)) = \psi(H(t, x)) \quad \text{or}
\]

In the nonlinear case, one obtains very different notions of such equivalence depending on the regularity hypotheses places on the map \( H \). At the minimum, one allows all homeomorphisms (that is continuous bijections with continuous inverses). At the other end one demands that, in addition, the map \( H \) is analytic (with analytic inverse).

In the case that \( H \) is continuously differentiable, the first equation yields the alternative description on the level of mapping vector fields to vector fields obtained by differentiating and evaluating at time equal to zero:

\[
\forall x \in U, \quad (DH)(x) f(x) = g(H(x))
\]

Compare this to the linear theory where the map \( H \) is given by matrix multiplication \( H(x) = Q x \). Locally, the only interesting points are singularities, since one easily establishes the following theorem:

**Theorem.** Suppose \( U \subset \mathbb{R}^n \) open, \( x_0 \in U \), and \( f : U \mapsto \mathbb{R}^n \) is of class \( C^r \). Then there exists a \( C^r \) map defined on some neighborhood of \( x_0 \) that maps \( x_0 \) to 0 and transforms the system \( x' = f(x) \) into the system \( y'_1 = 1, y'_2 = 0, \ldots, y'_n = 0 \).

Since linear systems are so much more amenable to complete analysis it is a natural question to ask when a nonlinear systems is really a linear system in disguise, i.e. only presented in the wrong set of coordinates.
**Hartman-Grobman-Theorem.** Suppose \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is of class \( C^1 \), and \( 0 \in \mathbb{R}^n \) is a hyperbolic fixed point of \( f \), i.e., \( f(0) = 0 \), and \( (Df)(0) \) has no purely imaginary eigenvalues. Then there exists an open neighborhood \( U \) of the origin and a homeomorphism \( H: U \rightarrow \mathbb{R}^n \) such that for all \( x \in U \)

\[
(H \circ f)(x) = (Df)(0) \cdot H(x).
\] (4)

The original proofs due to Grobman and Hartman date to 1959 and 1960. The latter may also be found in the textbook by Perko, and it is based on a successive approximation scheme. It is noteworthy that even if \( f \) is of class \( C^\infty \), one can in general only guarantee the existence of a homeomorphism, i.e. \( H \) being of class \( C^0 \).

It is instructive to consider topological conjugacy even in the case of pairs of linear systems, where one can write out explicit formulas for the homeomorphism. The slide [http://math.asu.edu/~kawski/classes/mat475/handouts/hsd-4-5.pdf](http://math.asu.edu/~kawski/classes/mat475/handouts/hsd-4-5.pdf) follows an example from the textbook by Hirsch, Smale, and Devaney.

**Exercise.** Demonstrate that two smooth vector fields \( f \) and \( g \) with \( f(0) = g(0) = 0 \) can be \( C^1 \) conjugate (in a neighborhood of \( 0 \) only if \( (Df)(0) \) and \( (Dg)(0) \) have the same eigenvalues (each with the same multiplicity). In other words, the eigenvalues of the Jacobian at the singularity form an invariant under conjugation by diffeomorphisms.

A different approach was taken by Poincaré who considered systems and coordinate changes defined by formal power series. The basic idea is that if \( f \) is analytic, and \( f(0) = 0 \), then one may try to use an infinite sequence of polynomial coordinate changes to successively eliminate all nonlinear terms, and in the limit achieve a coordinate change that transforms \( x' = f(x) \) into the linear system \( y' = (Df)(0)y \). The main result on *Poincaré normal forms* guarantees that under mild *nonresonance conditions* on the eigenvalues of \( (Df)(0) \) this is indeed possible. The details, and the convergence issues of this infinite composition of formal coordinate changes are beyond the scope of our course. However, the strategy of the approach, the emergence of the resonances, and the utility of a finite simplification are all relevant to us. In particular, even in the case of resonance, the possible normal forms are important starting points for classical bifurcation analysis. In other applications it is very helpful to be able to guarantee that a system can be approximated by a linear system up to order \( \|x\|^p \) with \( p \) as large as desired – solutions of the linear system are calculated and analyzed exactly, and this way one can quantify the size of the error.

In the sequel we disregard formal hypotheses, and take an exploratory approach, and add hypotheses as we go, see how far we can go, and what the merging obstacles are. See the MAPLE implementation [http://math.asu.edu/~kawski/MAPLE/475/08-02-19-poincareNF.mws](http://math.asu.edu/~kawski/MAPLE/475/08-02-19-poincareNF.mws) for illuminating calculations.

Start with a vector field \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) that is as smooth as needed – \( C^p \) for a finite improvement, or analytic for the original theorem. Assume that \( f(0) = 0 \) and write \( J = (Df)(0) \). Without loss of generality we may assume that \( J \) is in Jordan canonical form (else perform a constant linear change of coordinates). Expand \( f \) into a Taylor series, grouping together terms of the same order of homogeneity, i.e.

\[
f(x) = Jx + \sum_{k=2}^{\infty} f_k(x) \quad \text{where } \forall k \geq 2, \ f_k: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ satisfies } \forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^n, \ f_k(\lambda x) = \lambda^k f(x) \quad (5)
\]

The strategy is to use a sequence of *near identity transformations* that successively *kill* terms of order \( k \), for each \( k \geq 2 \). For the first stage, formally try a smooth map \( h_2: \mathbb{R}^n \rightarrow \mathbb{R}^n \) that is such that \( \forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^n, \ h_2(\lambda x) = \lambda^2 h(x) \) and write

\[
x = y + h_2(y) \quad (6)
\]

Differentiating this coordinate change along solutions of \( x' = Jx + f_2(x) + o(|x|^3) \) yields for terms of order less or equal to 2

\[
J(y + h_2(y)) + f_2(y + h_2(y)) + \ldots = (\text{Id} + (Dh)(y)) \cdot y' \quad (7)
\]

For \( y \) sufficiently close to 0, the linear map \( (\text{Id} + (Dh)(y)) \) is invertible, and for terms of order less than 3
Putting this together with the desired result \( y' = Jy + \ldots \), with no terms of order 2, we obtain
\[
(Id - (Dh)(y))J(y + h_2(y)) + f_2(y + h_2(y)) + \ldots = Jy + o(|y|^2)
\]
(9)

Expanding the left hand side, and only keeping terms of order less or equal to 2 yields
\[
Jy + Jh_2(y) - (Dh)(y) \cdot y + f_2(y) = Jy + o(|y|^2)
\]
(10)
leading to the homological equation
\[
(Dh)(y) \cdot y - Jh_2(y) = f_2(y).
\]
(11)

If for given \( f_2 \) one can find a homogeneous (quadratic) \( h_2 \) that satisfies this equation, then the transformed system will have no quadratic terms.

Consider the vector space of all quadratic vector fields (or quadratic transformations)
\[
V_2 = \{ h: \mathbb{R}^n \to \mathbb{R}^n: h \in C^\omega \text{ and } \forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^n, h(\lambda x) = \lambda^2 h(x) \}.
\]
(12)
and observe that the operator
\[
L_2: h \mapsto L_2(h)(y) = (Dh)(y) \cdot y - Jh_2(y)
\]
is linear and it maps \( V_2 \) into itself. (Verify this!) But \( V_2 \) is a finite dimensional vector space. (Find its dimension, and also the dimension for the cubic, quartic analogues!). Thus one can solve the homological equation for any given \( f_2 \) if and only if the nullspace of \( L_2 \) is trivial. This is a good time to continue the exploration using computer algebra. In particular, one readily discovers the nonresonance conditions.

It takes substantially more work that the nonresonance conditions are also sufficient for being able to carry out an infinite sequence of such transformations, and to show that this sequence converges to an analytic coordinate change on an open neighborhood of the singularity (e.g. initially one needs to to be worried that at each consecutive coordinate change is only valid on a smaller neighborhood). We will stop here with the theoretical work, and explore specific examples using computer algebra.

Note that for bifurcation and normal form theory, it is also useful to consider the case when the \( L_2 \) is not injective. In that case one can kill all terms in \( f_2 \) that lie in the range of \( L_2 \). Consequently, the only terms one may have to contend with are those that lie in a subspace complementary to this range. One has substantial latitude to choose suitable bases for that complementary subspace and these define the classical normal forms studied in bifurcation theory.