1. Suppose \( f: \mathbb{C}^1(\mathbb{R}^n, \mathbb{R}^n), f(0) = 0 \). Write \( A = (Df)(0) \). Suppose \( \forall \lambda \in \sigma(A), \Re \lambda < 0 \). Using the inverse Lyapunov theorem, one may explicitly construct symmetric, positive definite matrices \( P, Q \in \mathbb{R}^{n \times n} \) such that \( A^TP + PA = -Q \). Note that \( \sigma(P) \cup \sigma(Q) \subseteq (0, \infty) \). Define \( 0 < \alpha = \max \sigma(P) \) and \( 0 < \beta = \min \sigma(Q) \). Note that for all \( x \in \mathbb{R}^n, x^TQx \leq \beta \|x\|^2_2 \) and \( \|Px\|^2_2 \leq \alpha \|x\|^2_2 \) and Fix any positive \( \varepsilon < \frac{\beta}{2\alpha} \). Since \( f \in \mathbb{C}^1 \) and \( (Df)(0) = A \), there exists \( \delta > 0 \) such that for all \( x, y \), if \( \|x\|_2 < \delta \) then \( \|f(x) - Ax\|_2 < \varepsilon \|x\|_2 \). Define \( V: \mathbb{R}^n \mapsto \mathbb{R} \) by \( V(x) = x^TPx \). For any \( x \in \mathbb{R}^n \) such that \( 0 < \|x\|_2 < \delta \), calculate

\[
\dot{V}(x) = \nabla V \cdot f(x) = -x^TQx + \nabla V \cdot (f(x) - Ax)
\]

\[
\leq -\beta \|x\|^2_2 + 2\|Px\|_2 \cdot \|f(x) - Ax\|_2
\]

\[
\leq -\beta \|x\|^2_2 + 2\alpha \|x\|^2_2 \cdot \varepsilon \|x\|_2 < 0.
\]

This shows that \( V \) is a strict Lyapunov function for \( x' = f(x) \) on the open \( \delta \)-ball about the origin, which hence is a locally asymptotically stable equilibrium point of \( x' = f(x) \). Denoting by \( c = \min\{V(x): \|x\|_2 = \delta\} \), the basin of attraction contains the open set \( V^{-1}(c) \).

2. A system \( x' = f(x) \) on \( \mathbb{R}^n \) is a gradient system if there exists a differentiable function \( V: \mathbb{R}^n \mapsto \mathbb{R} \) such that for each \( 1 \leq i \leq n \), \( \frac{\partial V}{\partial x_i} = f_i \). If \( V \) is twice continuously differentiable this implies that for all \( i, j \leq n \), \( \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \). On a simply connected domain, this condition is also sufficient. A system \( x' = f(x) \) on \( \mathbb{R}^{2n} \) with canonical coordinates \( x = (q, p) \) is Hamiltonian if there exists a differentiable function \( H: \mathbb{R}^{2n} \mapsto \mathbb{R} \) such that for each \( 1 \leq i \leq n \), \( f_i = \frac{\partial H}{\partial p_i} \) and \( f_{n+i} = -\frac{\partial H}{\partial q_i} \). If \( H \) is twice continuously differentiable this implies that for all \( i, j \leq n \), \( \frac{\partial f_i}{\partial q_j} + \frac{\partial f_j}{\partial p_i} = 0 \). Again, on a simply connected domain this condition is also sufficient.

For a planar system \( \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \) these conditions translate into \( b = c \) and \( a + d = 0 \), respectively. Possible potential and Hamiltonian functions are \( V(x, y) = \frac{1}{2}(ax^2 + dy^2) + bxy \) (if \( b = c \)), and \( H(x, y) = \frac{1}{2}(ax^2 - by^2) + axy \) (if \( a + d = 0 \)). Any \( 2 \times 2 \)-matrix can always be written (in many different ways) as a sum of a symmetric and trace-zero matrix. A natural choice is to write \( Ax \) as the sum of \( \frac{1}{2}(A + A^T)x \) and \( \frac{1}{2}(A - A^T)x \) which are gradient and Hamiltonian, respectively. In the case of \( 2n = 4 \), the above relations translate into \( b = c^T \) and \( a + d^T = 0 \) for \( a, b, c, d \in \mathbb{R}^{2 \times 2} \). The example where all entries \( A \in \mathbb{R}^{4 \times 4} \) are zero except \( A_{12} = 1 \) shows that in general it is impossible to satisfy these conditions at the same time. [In general, one defines Hamiltonian systems in a coordinate-free way that does not require that the first and second half of the coordinate vector are the states and momenta. There are also important systems which are both gradient and Hamiltonian!]

3. Apply the conditions of the preceding exercise, and integrate where possible:

(i) \( x' = y^2 + 2xy, \ y' = x^2 + 2xy \), gradient of \( V = x^2y + xy^2 \), not Hamiltonian.
(ii) \( x' = x^2 - 2xy, \ y' = y^2 - 2xy \), not gradient, Hamiltonian \( H = x^2y - xy^2 \).
(iii) \( x' = x^2 - 2xy, \ y' = y^2 - x^2 \), gradient of \( V = \frac{1}{2}(x^3 + y^3) - x^2y \), not Hamiltonian.
(iv) \( x' = y, \ y' = x - x^2 \), not gradient, Hamiltonian \( H = \frac{1}{2}(y^2 - x^2) + \frac{1}{4}x^4 \).

4. Let \( \phi, \psi: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n \) denote the general solutions of the systems \( x' = Ax \) and \( x' = Ax + g(t) \), respectively. I.e., for all \( t, t_0 \in \mathbb{R} \) and all \( y_0 \in \mathbb{R}^n \), \( \phi(t, t_0, y_0) = \psi(t, t_0, y_0) = y_0 \) and \( \frac{d}{dt} \phi(t, t_0, y_0) = A\phi(t, t_0, y_0) \) while \( \frac{d}{dt} \psi(t, t_0, y_0) = A\psi(t, t_0, y_0) + g(t) \). By linearity, for all \( t, t_0 \in \mathbb{R} \) and all \( y_1, y_2 \in \mathbb{R}^n \), \( \psi(t, t_0, y_1) - \psi(t, t_0, y_2) = \phi(t, t_0, y_1) - \phi(t, t_0, y_2) = \phi(t, t_0, y_1 - y_2) = \phi(t, t_0, y_1 - y_2) - \phi(t, t_0, 0) \). In particular, two solutions (of either system) satisfying
initial conditions \(y(t_0) = y_1\) and \(y(t_0) = y_2\) are at time \(t\) within \(\varepsilon > 0\) of each other if and only if the
the solution of the inhomogeneous system satisfying the initial condition \(y(t_0) = y_1 - y_2\) is at time
\(t\) within \(\varepsilon\) of the zero solution of the homogeneous system. Hence (asy) stability of any solution of
either system is equivalent to (asy) stability of the zero solution of the homogeneous system.
In particular, the solution \(\psi(t, 0, 3) = 3e^{2t}\) of \(y' = -y + 9e^{2t}\) is asymptotically stable. Pictorially
this means that as \(t \to \infty\) every solution gets and remains arbitrarily close to this solution (while,
of course, this, and all other solutions grows beyond every finite bound as \(t\) grows.)

5. A nontrivial periodic solution is never asymptotically stable. After any integer multiple of the
period the solutions starting at any two distinct points on the periodic orbit will return to these
points, and in particular have the same distance from each other that they had initially.

The phase plane portrait of \(r' = r(1 - r), \theta' = 1\) (in polar coordinates) suggests that the periodic
orbit \(r = 1\) be considered asymptotically stable – but the periodic solutions do not satisfy the
criterion given in the definition. We may propose a different notion of orbitally stable: The solution
curve \(\phi(\cdot, t_0, y_0)\) is locally orbitally stable if there exists an open set \(\Omega \subseteq \mathbb{R} \times \mathbb{R}^n\) containing \((t_0, y_0)\)
such that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that
and every \((t_1, y) \in \Omega\), if \(\|\phi(t, y) - \phi(t_1, y)\| < \delta\)
then for every \(t \geq t_1\) there exists \(\tau \geq t_0\) such that \(\|\phi(t, y) - \phi(\tau, y_0)\| < \varepsilon\).
It is locally asymptotically orbitally stable if in addition for every \(\varepsilon > 0\) and every \((t_1, y) \in \Omega\) there exists
\(T \in \mathbb{R}\) such that for all \(t\), if \(t \geq T\) then there exists \(\tau \in \mathbb{R}\) such that
\(\|\phi(t, y) - \phi(\tau, y_0)\| < \varepsilon\). Clearly one can come up with a myriad of similar definitions. These generally do not have universally
agreed-upon names.

6. Suppose \(f: \mathbb{R}^2 \mapsto \mathbb{R}^2\) is continuously differentiable, \(f(0) = 0\) and \(x' = f(x)\) is a Hamiltonian system.
Writing \(x = (q, p)\), \(f_1 = \frac{\partial H}{\partial p}\) and \(f_2 = -\frac{\partial H}{\partial q}\), the characteristic polynomial of \(A = (Df)(0)\) is

\[
p_A(s) = \left( s^2 - \left( \frac{\partial^2 H}{\partial q \partial p} \right)^2 \right) + \frac{\partial^2 H}{\partial q^2} \cdot \frac{\partial^2 H}{\partial p^2}
\]
whose roots clearly lie on the real or imaginary axes.

Denote by \(J\) the \(2n \times 2n\) matrix consisting of two \(n \times n\) blocks of zero matrices along the diagonal, and
whose off diagonal blocks are the \(n \times n\) identity matrix and its negative, (i.e. \(J_{ij} = \pm 1\) if \(j - i = \pm n\)
and \(J_{ij} = 0\) otherwise). For a Hamiltonian vector field \(f = \left( \frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right) = J \left( \frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right)\) the criterion
discussed in item 2 about equality of mixed partial derivatives yields for \(A = (Df)(0) = (D^2 H)(0)\)
the characterization: \(J \cdot A \cdot J = A^T\). (Think about what left and right multiplication by \(J\) do to
the rows and columns of a matrix, and which blocks of \((Df)(0)\) must be equal to (or transposes,
negatives of) each other if \(f\) is a Hamiltonian vector field.) Using that \(J^T = -J\), we find for the
characteristic polynomial of the Jacobian \(A = (Df)(0)\) of a Hamiltonian vector field

\[
\text{det}(\lambda I - A) = \text{det}((\lambda I - JAJ)^T) = \text{det}(\lambda I - JAJ) = \text{det}(J \cdot (-\lambda) I \cdot J + JAJ)
= \text{det}(J) \text{det}((-\lambda) I - A) \cdot \text{det}(J) = (\text{det}(J))^2 \text{det}((-\lambda) I - A).
\]

Hence \(\lambda\) is an eigenvalue of \((Df)(0)\) iff \(-\lambda\) is an eigenvalue of \((Df)(0)\). Together with the fact
that eigenvalues of real matrices appear in complex conjugate pairs we conclude that eigenvalues of
Hamiltonian vector fields that are neither real nor purely imaginary, always occur in quadruplets
\(\pm a \pm bi\) (all combinations of signs).

In the case that \(x' = f(x)\) is a gradient system \((Df)(0)\) is symmetric and hence all eigenvalues are
real. Conversely, for any set of real numbers \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\), the vector field \(f(x) = (\lambda_1 x_1, \ldots, \lambda_2 x_n)\)
is clearly a gradient system, and hence nothing else can be said about the zeros of the Jacobian of
a general gradient system.