Explain what you are doing. Computer printouts without explanations and formulas scattered over a page without clear logical order will be ignored (zero credit!) It is YOUR responsibility to demonstrate that you have mastered the material of this class. Check ALL results with available computer software!

1.a. Use separation of variables and Fourier expansions to solve the (PDE) \( u_{tt} = u_{xx} \) in the domain \( 0 < x < \pi \) and \( 0 < t \), with boundary conditions (BC1,BC2) \( u(0,t) = u(\pi, t) = 0 \) for all \( t \geq 0 \), (BC3) \( u_t(x,0) = 0 \) for all \( 0 \leq x \leq \pi \), and

\[
(C4) \quad u(x,0) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{\pi}{3} \\
 x - \frac{\pi}{3} & \text{if } \frac{\pi}{3} \leq x \leq \frac{2\pi}{3} \\
 2\pi - x & \text{if } \frac{2\pi}{3} \leq x \leq \pi
\end{cases}
\]

In this exercise only: Explanation may be kept very short. The emphasis is to demonstrate that you can calculate effectively. Simplify only as far as needed so that e.g. a finite Fourier approximation may be evaluated, plotted, or animated easily.

b. Write out explicitly a fifth order Fourier approximation of the solution \( u(x,t) \) (with four-digit decimal approximations of the Fourier coefficients).

c. Sketch the graph of \( u(x,t_i) \) (as a function of \( x \)) for several values of \( t_i \), e.g. \( t_i = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \).

You may use a Fourier approximation (as in part b. instead of the exact solution.

d. What physical phenomena / objects are modeled by this (PDE) and (BCs)? Is (BC4) a reasonable boundary condition (that is easily created in an experimental set-up)? (E.g. “how many fingers does it take” to produce these (BCs)? – Explain how!)

**Bonus**: Graphically demonstrate how the solution \( u(x,t) \) can be considered the superposition of two traveling waves – i.e. sketch these two as functions of \( x \) for several values of \( t \).

2.a. Classify the (PDE) \( u_t - u_{xx} = 0 \) as hyperbolic, elliptic or parabolic.

b. Is this a linear partial differential equation? Why?

c. Briefly describe a physical phenomenon modeled by the (PDE).

In terms of this physical scenario, and understandable by first-year calculus students, explain the choice of the sign – i.e. why the model is \( u_t - u_{xx} = 0 \), and **not** \( u_t + u_{xx} = 0 \).

d. Continuing with this physical application, what do the boundary conditions (BC12) \( u(0,t) = u(L,t) = 0 \) and (BC12’) \( u_x(0,t) = u_x(L,t) = 0 \), respectively, mean in practical terms?

3. Consider the two-dimensional wave equation (PDE) \( u_{tt} = u_{xx} + u_{yy} \) on the square \( S \) given by \( 0 \leq x \leq \pi, 0 \leq y \leq \pi \) with boundary conditions (BC1,BC2) \( u(x,0,t) = u(x,\pi, t) = 0 \), for all \( 0 \leq x \leq \pi, 0 \leq t \) and (BC3,BC4) \( u(0,y,t) = u(\pi,y, t) = 0 \) for all \( 0 \leq y \leq \pi, 0 \leq t \).

a. Find all values of \((m,n,k)\) for which \( u_{mnk}(x,y,t) = \sin mx \sin ny \cos kt \) is a solution of (PDE) together with the boundary conditions (BC1)-(BC4). Carefully explain **how** each of (PDE) and the (BCs) places constraints on **which** of \( m, n, \) and \( k, \) respectively.

b. Verify directly that (or use theory to explain why) \( w = (2 \sin 5x \sin 5y - 3 \sin 7x \sin y) \cdot \cos \sqrt{50}t \) is a solution of (PDE) with (BC1) - (BC4).