1. Let \((x, y)\) and \((r, \theta)\) denote rectangular and polar coordinates in the plane (i.e. \(x = r \cos \Theta\) and \(y = r \sin \Theta\)). Express the partial derivatives \(u_{rr}\) and \(u_{\Theta\Theta}\) of a function \(u\) in terms of partial derivatives with respect to \(x\) and \(y\).

2. Put the Hermite-ODE \(y'' - 2xy' + 2\lambda y = 0\) into Sturm-Liouville form \((py')' + (q + \lambda r)y = 0\). Hint: Find an “integrating factor” \(p(x)\).

3. Find all eigenfunctions and eigenvalues \(\lambda\) of the Sturm-Liouville problem \(y'' + \lambda^2 y = 0\) with (BC1) \(y(0) = 0\) and (BC2) \(y'(\pi) = 0\).

4. Consider the special case of a Sturm Liouville problem \((xy')' + \lambda x^2y = 0, y(0) = y(1) = 0\).
   a. Give a precise statement regarding the orthogonality of the eigenfunctions.
   b. Prove their orthogonality in this special case.

5. Briefly explain why/how orthogonality is so useful for solving PDEs.

**Work only one of the following two problems. [[I will grade only one!]]**

6. Let \(f\) be the \(2\pi\)-periodic function defined by \(f(t) = \frac{\pi}{2} - |t|\) if \(|t| \leq \frac{\pi}{2}\) and \(f(t) = 0\) if \(\frac{\pi}{2} \leq |t| \leq \pi\). Consider the (PDE) \(0 = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\Theta\Theta}\) on the unit disk \(r < 1\), subject to the boundary condition (BC1) \(u(1, \Theta) = f(t)\) for \(-\pi \leq \Theta < \pi\).
   a. Sketch the graph of the function \(f\).
   b. Give a brief description of a physical (biological, …) situation/process that might be modelled by (PDE). Briefly discuss the practical interpretation of (BC1) and (BC2).
   c. Expand \(f\) into a Fourier series – provide both a simplified formula for the Fourier coefficients and numerical approximations of the first five nonzero coefficients.
   d. Separate variables and find the eigenfunctions and eigenvalues.
   e. Write the solution of the boundary value problem as an infinite series with simplified formula for its general term.
   f. Write a five-term approximate solution with four digit decimal approximations of the coefficients. Plot this approximate solution.

6'. Consider the (PDE) \(u_t = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\Theta\Theta}\) for \(r < 1\), \(0 < t\), subject to (BC1') \(u(1, \Theta, t) = 0\) for \(-\pi \leq \Theta < \pi\), \(t > 0\), and (BC2') \(u(r, \Theta, 0) = r^2(1-r)(1+\cos 2\Theta)\) for \(-\pi \leq \Theta < \pi\), \(r < 1\).
   a. Give a brief description of a physical (biological, …) situation/process that might be modelled by (PDE'). Briefly discuss the practical interpretation of (BC1') and (BC2').
   b. Separate variables and find the eigenfunctions and eigenvalues.
   c. Expand the boundary datum (BC2') in an appropriate series. Provide a simplified formula for the general coefficient, and approximate numerical values for the first five nonzero terms.
   d. Write the solution of the boundary value problem as an infinite series with simplified formula for its general term.
   e. Write a five-term approximate solution with four digit decimal approximations of the coefficients. Plot this approximate solution.