Justify all major steps that involve substantial complex analysis reasoning. On the other hand, there is no need for lots of detail in steps that involve only calculus or algebra – often a computer print-out may be adequate documentation. You may use MAPLE throughout, but it is YOUR responsibility to demonstrate that you have mastered the new material of this class.

1. a. Write the number $(1 + i\sqrt{3})^4$ in the form $a + bi$ with $a$ and $b$ real.
   b. Find all solutions of the equation $z^3 = i$.
   **Bonus:** Find all solutions of the equation $z^i = i$.

2. a. Describe the set $A = \{z \in \mathbb{C} : 2 \leq \Re(z) \leq 3 \text{ and } 0 \leq \Im(z) \leq \pi \}$ in words, and sketch it.
   b. Describe the image $f(A)$ of the set $A$ under the map $f : z \mapsto e^z$.
      Carefully explain why $f$ maps onto the set you claim to be $f(A)$.
   c. Show that $f$ maps the region $A$ one-to-one onto the set $f(A)$ you found in part b.

3. a. Show that if $f$ is analytic, then its real part $u = \Re(f)$ is harmonic.
    You may assume (we will prove this soon) that the real and imaginary parts of every analytic function have continuous derivatives of all orders.
    b. Show that if $u$ and $v$ are conjugate harmonic functions, then the product $uv$ is harmonic.
       **Hint:** For an elegant argument consider $\Im ((u + iv)^2)$.
    **Bonus:** Give an explicit counterexample to show that the product of two harmonic functions need not be harmonic.
   c. Demonstrate that $u : (x, y) \mapsto \log \sqrt{x^2 + y^2}$ is harmonic, and find all its harmonic conjugates.

4. a. Find the Jacobian matrix of partial derivatives of $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by $F(x, y) = (x^2, -y^2)$.
   b. Explain in elementary terms (e.g. using the definition of differentiability) why the function $f : \mathbb{C} \mapsto \mathbb{C}$ defined by $f(z) = (\Re(z))^2 - i (\Im(z))^2$ is not analytic, even though $F$ is (real) differentiable.
   c. Show that if a function mapping $\mathbb{C}$ to $\mathbb{C}$ is (complex) differentiable at a point $z_0$ then its real and imaginary parts satisfy the Cauchy-Riemann equations at $z_0$.
   d. Find all points $z$ at which the function $f$ of part a. satisfies the Cauchy-Riemann equations.
   e. Find all points where the function $f$ is analytic.

5. a. Use the definition of the complex cosine (in terms of the complex exponential) to derive the expression $-i \log (z + \sqrt{z^2 - 1})$ for $\cos^{-1}(z)$ in terms of the complex logarithm.
   b. Find all values of $\cos i$. 