Justify all major steps that involve substantial complex analysis reasoning. On the other hand, there is no need for lots of detail in steps that involve only calculus or algebra – often a computer print-out may be adequate documentation. You may use MAPLE throughout, but it is YOUR responsibility to demonstrate that you have mastered the new material of this class.

1. Let \( f(z) = \frac{1}{(z^2 + 1)^2} \). (If you prefer you may work with \( F(z) = \frac{1}{(z^2 + 1)(z^2 + 2z + 5)} \) instead.)
   a. Use the residue theorem to evaluate the (real, improper) integral \( \int_{-\infty}^{\infty} f(x) \, dx \).
      Explain all necessary steps in detail – include at least the following:
      (i) Find all singularities of \( f(z) \) (considered as a function of a complex variable \( z \)).
      (ii) Describe a parameterized family of closed contours suitable for using the residue theorem.
      (iii) Provide detailed estimates of the magnitude the integral over the added curves.
      (iv) Calculate the residues of \( f(z) \) at all relevant singularities.
      (v) Apply the residue theorem, and explain in detail what happens in the limit as the contours . . .
   b. Find the first three nonzero terms of the Laurent series expansion of \( f(z) \) about \( z = i \).
   c. Exhibit the disk of convergence of the Laurent series expansion of \( f(z) \) about \( z = i \).
   d. Let \( C \) be the circle with center \( i \) and radius \( R = 1 \) oriented counterclockwise.
      Evaluate the integral \( \int_C f(z) \, dz \) directly (via parameterizations) or via Cauchy’s integral formula.
      Explain how you could have evaluated the integral in part a. without reference to the residue theorem – outline the general procedure involving Cauchy’s theorem and deformation of contours in this particular case.

2.a. Consider \( u: \mathbb{R}^2 \mapsto \mathbb{R} \) defined by \( u(x, y) = x^2 \).
      Find a function \( v: \mathbb{R}^2 \mapsto \mathbb{R} \) such that \( u + iv \) is analytic. If impossible, explain why.
   b. Consider \( u: \mathbb{R}^2 \setminus \{0\} \mapsto \mathbb{R} \) defined by \( u(x, y) = \ln \left( \sqrt{x^2 + y^2} \right) \).
      (i) Find a function \( v: \mathbb{R}^2 \mapsto \mathbb{R} \) such that \( f = u + iv \) is analytic. If impossible, explain why.
      (ii) Identify a maximal domain for this function \( f \). Briefly explain the problems.
      (iii) Calculate and sketch the images of the following curves under the map \( w = f(z) \):
      The “upper” semicircles with center \( 0 \), and radii \( 1, 2 \) and \( 3 \).
      The segments of the lines \( x = 0 \), \( y = 0 \) and \( y \pm x \) that lie on or above the real axis and between
      the circles \( |z| = 1 \) and \( |z| = 3 \).