1.a. \( \vec{N} \cdot (\vec{R} - \hat{A}) = 0 \). Here \( \vec{N} = 4\hat{i} + 2\hat{j} + \hat{k} \), e.g. as a multiple of the cross-product \((\vec{B} - \hat{A}) \times (\vec{C} - \hat{A})\), c.f. MAPLE scratch paper.

b. \( \vec{P} = \vec{Q} + \frac{(\vec{A} - \vec{Q}) \cdot \vec{N}}{\vec{N} \cdot \vec{N}} \). (Geometrically, \( P \) is the intersection point of the the plane \( \Pi \) and the line through \( Q \) that is perpendicular to \( \Pi \). The calculation utilizes the projection of a vector from \( Q \) to any point on the plane onto the line perpendicular to the plane.)

c. \( \min \left\{ \left( x - 1 \right)^2 + \left( y - 2 \right)^2 + \left( z - 3 \right)^2 \right\} \) subject to \( 4x + 2y + z - 12 = 0 \).

Form the Lagrangian \( F = f + \lambda g \) and solve the system \( \{ \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = 0 \} \).

Written out this becomes
\[
\begin{align*}
2(x - 1) + 4\lambda &= 0 \\
2(y - 2) + 2\lambda &= 0 \\
2(z - 3) + \lambda &= 0 \\
4x + 2y + z &= 0
\end{align*}
\]

This is a linear system, i.e. straightforward to solve.

b. By definition \( \frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \) if the limit exists. But \( \frac{1}{h} \int_{h=0}^{h=0} \frac{f(h,0) - f(0,0)}{h} \) has no limit as \( h \to 0 \). c. Thus \( f \) is differentiable everywhere except at \((0,0)\).

(Note, \( f \) is not even continuous at \((0,0)\) since \( \lim_{x \to 0} f(x,0) = 4 \) whereas \( \lim_{y \to 0} f(0,y) = -2 \).

3.a. Using polar coordinates, the edges lie on the lines \( \theta = 0, \theta = \frac{\pi}{2} \) and \( r \cos \theta = 3 \). The area of \( T \) is \( A = \frac{9}{2} \). Thus the average distance is \( \bar{r} = \frac{1}{A} \int_0^{\pi/2} r \cdot r \, d\theta = (\sqrt{2} + \ln(1 + \sqrt{2}) \approx 2.295.

b. The faces – each defined by three vertices – of the pyramid lie on the planes \( x = 0, y = 1, z = x \), and \( z = y \) Thus the moment of inertia is \( I_z = \int_{x=0}^{1} \int_{y=x}^{y} \frac{2}{3} (y^2 + y^2) \, dz \, dy \, dx = \frac{7}{60} \delta \).

4. \( F(x,y) = y^2 \ddot{r} \) is not linear, is divergence free, is not irrotational, is not conservative. Thus there is no potential. \( G(x,y) = -(x\dddot{r} + y\dddot{r})/(x^2 + y^2) \) is not linear, is divergence free, is irrot., is conservative and \( \varphi(x,y) = -\sqrt{x^2 + y^2} = -\frac{1}{2} \ln(x^2 + y^2) \) is a potential. \( H(x,y) = x\dddot{r} + y\dddot{r} \) is linear, is divergence free, is irrot., is conservative and \( \varphi(x,y) = xy \) is a potential.

5.a. \( \vec{\nabla} \cdot (f \vec{F}) = \frac{\partial f}{\partial x} F_1 + \frac{\partial f}{\partial y} F_2 + \frac{\partial f}{\partial z} F_3 + \frac{\partial f}{\partial x} F_2 + \frac{\partial f}{\partial y} F_3 + \frac{\partial f}{\partial z} F_1 = \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) \cdot \vec{F} + f \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = (\vec{\nabla} f) \cdot \vec{F} + f (\vec{\nabla} \cdot \vec{F}).

b. \( \nabla (g^m) = g'(\varrho) \nabla \varrho = g'(\varrho) \frac{1}{\varrho} \vec{R} \cdot \vec{R} \). c. \( \nabla (g^m \vec{R}) = \left( m \varrho^{m-1} \frac{1}{\varrho} \vec{R} \right) \cdot \vec{R} + g^m \nabla \cdot \vec{R} = (m + 3) g^m \)

This is zero if and only if \( m = 3 \), compare problem 6.

Details for \( \frac{\partial}{\partial x} g \sqrt{x^2 + y^2 + z^2} \ddot{r} + \ldots = g'(\sqrt{\ldots}) \cdot \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} \ddot{r} + \ldots = g'(\varrho) \cdot \frac{2\varrho}{2\varrho} \ddot{r} + \ldots \).

6.a. Gravitational and electric field of a point mass or charge.

b. The vector field is of constant magnitude \( ||E|| = \frac{1}{\varrho^2} \) on \( S \) and is perpendicular to the surface \( S \). Hence the surface integral equals its magnitude times the area of the surface: \( \int_S \vec{E} \cdot \vec{N} \, dA = \frac{1}{\varrho^2} \cdot \frac{1}{8} 4\pi^2 = \frac{\pi}{2} \) (which is independent of \( a \)).

c. Consider the region in 3-space that is bounded by the piece \( S \) of the sphere in part b., e.g. with radius \( a = 1 \), the triangle \( T \) of problem 1 and parts of the coordinate planes. Since the divergence of \( \vec{E} \) is identically zero away from the origin, i.e. at all points in this region, by Gauss’ divergence theorem the total flux across the boundaries of this region is zero. Since the vector field is tangent to the coordinate planes, the flux across the parts of the boundary of \( R \) that lie on these planes is zero. Thus, taking into account the original orientations of \( S \) and \( T \), the flux across \( T \) must equal the flux across \( S \), i.e. \( \int_T \vec{E} \cdot \vec{N} \, dA = \frac{\pi}{2} \).