1. Suppose $R, R'$ are commutative rings, $N \triangleleft R$ is an ideal, and $\Phi : R \rightarrow R'$ is a ring-homomorphism.
   a. Show that $\Phi(N)$ is an ideal of $\Phi(R)$.
   b. Give a counterexample to show that $\Phi(N)$ need not be an ideal of $R'$.

2. a. Show that there is no simple group of order 12.
   b. Suppose that $p$ and $q$ are primes with $q < p$. Show that there is no simple group of order $p^2q$.
      (For bonus credit only assume that $p$ and $q$ are distinct odd primes.)

3. Suppose $X$ is a set and $Y$ is a group, a ring, integral domain, or a field. What can you say about the space of functions $Y^X = \{f : X \rightarrow Y\}$ with operations defined pointwise $(f * g)(x) \equiv f(x) * g(x)$ in each case? Briefly explain which properties are inherited, and give counterexamples to demonstrate that other properties are not automatically inherited.

4. a. Write $f(x) = x^9 - x \in \mathbb{Z}_3[x]$ as a product of irreducible polynomials in $\mathbb{Z}_3[x]$.
   (Hint: Calculate $(x^2 + x - 1)(x^2 - x - 1)$ in $\mathbb{Z}_3[x]$.)
   b. Let $\alpha$ be a root of $x^2 + 1 \in \mathbb{Z}_3[x]$. Exhibit a basis for the extension field $\mathbb{Z}_3(\alpha)$ over $\mathbb{Z}_3$ and list all elements of $\mathbb{Z}_3(\alpha)$.
   c. Describe the additive and the multiplicative structures of the field $\mathbb{F}_9 \equiv \mathbb{Z}_3(\alpha)$ with 9 elements. Give generators of the (sub)groups in terms of the basis in b.
   d. Exhibit the relation between the irreducible factors of the polynomial $f(x)$ from a. and the elements of the field $\mathbb{F}_9 \equiv \mathbb{Z}_3(\alpha)$ of b.

5. In the group $\mathbb{Z}_{36}$ let $H = \langle 6 \rangle$ and $N = \langle 9 \rangle$.
   a. List the elements in $H + N$ and the elements in $H \cap N$
   b. List the cosets in $(H + N)/N$, showing the elements of each coset.
   c. List the cosets in $H/(H \cap N)$, showing the elements of each coset.
   d. Explicitly exhibit the map between $(H + N)/N$ and $H/(H \cap N)$ as in the 2nd isomorphism theorem.

6. Let $R$ be a ring and $M \triangleleft R$ and $N \triangleleft R$ be ideals.
   a. Show that $M + N$ is an ideal of $R$.
   b. Show that $\Phi : M + N \rightarrow M/(M \cap N)$ defined by $\Phi(m + n) = m + (M \cap N)$ is indeed well-defined.
   c. Conclude that the ring $(M + N)/N$ is naturally isomorphic to the ring $M/(M \cap N)$.

7. a. State the definitions of integral domain, principal ideal domain, Euclidean domain, and unique factorization domain.
   b. Partially order the properties from a. (e.g. $P1 > P2$ meaning “if $R$ is a $P1$ then $R$ is also a $P2$”). No proofs required.
   c. Give at least one example to show that at least of the orderings in b. is strict (e.g. if $P1 > P2$ exhibit an example that is $P2$ but not $P1$). No proofs required, but short comments welcome.