1. Suppose \((G, \cdot)\) is an associative binary structure with a left identity \(e\), in which every \(x \in G\) has a left inverse \(x' \in G\).
   a. Define left identity and left inverse.
   b. Show that “the left identity in \(G\) is unique”. (Hint: First analyze \(x''x'x\) for some \(x \in G\)).

2. a. Give a precise statement of the division algorithm.
   b. Outline a proof that every subgroup of a finite cyclic proof is cyclic.
   c. Suppose \(G\) is a cyclic group of order \(n \in \mathbb{Z}^+\) generated by \(a \in G\).
      Find the set of all generators of \(G\) – prove that you are correct.
   d. Describe (without proof) the set of all subgroups of a cyclic group of order \(n \in \mathbb{Z}^+\).

3. Suppose \(G\) is a finite abelian group, \(m \in \mathbb{Z}^+\), and \(\Phi \colon G \mapsto G\) is defined by \(\Phi(x) = x^m\).
   a. Show that \(\ker \Phi = \{x \in G : \Phi(x) = e\}\) and \(\Phi[G] = \{\Phi(x) : x \in G\}\) are subgroups of \(G\).
   b. Is (n)either or both of these necessarily a subgroup without the assumption of \(G\) being abelian?
      Give a counterexample or sketch an argument in each case.
   c. Describe conditions on \(m\), \(G\) under which either subgroup is trivial or improper (equal to \(\{e\}\) or \(G\)).

4. a. Suppose \(G\) is a finite group of order \(|G| = n\). Show that for every \(a \in G\), \(a^n = e\) (identity in \(G\)).
   b. Show that every group of order 6 has at least two elements of order 3.
      (Do not just refer to a list of all groups of order 6 unless you prove that your list is complete.)
   **Bonus.** Does your argument generalize to the case of \(|G| = 10\), or \(|G| = pm\) with \(p\) prime?

4. Suppose \(G\) is a group, \(a \in G\), and \(H \leq G\) is a subgroup.
   Which of the following are necessarily subgroups of \(G\). Briefly justify your answers.
   a. \(\{g \in G : gag^{-1} = a\}\)
   b. \(\{g \in G : ghg^{-1} = h \text{ for all } h \in H\}\)
   c. \(\{g \in G : ghg^{-1} \in H \text{ for all } h \in H\}\)
   d. \(\{g \in G : ghg^{-1} = h \text{ for all } h \in G\}\)
   **Bonus.** Which of those that are subgroups are necessarily abelian?
   **Bonus.** Give the standard names, if known, for those that are subgroups.

6. a. Under what conditions do two transpositions \(\tau, \tau' \in S_n\) commute? Prove your result.
   b. Show that every 3-cycle in \(S_n\) is a product of two transpositions in \(S_n\).
   c. Prove that every product of two transpositions in \(S_n\) be written as a (product of) 3-cycle(s) in \(S_n\).
   **Bonus.** Find the order of the subgroup of \(S_n\) that is generated by the set of all 3-cycles in \(S_n\).

7. Suppose \(H\) is a subgroup of a finite group \(G\). For \(g \in G\) define the cosets \(gH = \{gx : x \in H\}\)
   and \(Hg = \{xg : x \in H\}\). In each of the following prove the statement of give a counterexample.
   a. If \(a \in G\) and \(aH\) is a subgroup of \(G\) then \(a = e\) (identity in \(G\)).
   b. If \(a, b \in G\) and \(aH = bH\) then \(Ha = Hb\).
   c. If \(a, b \in G\) and \(aH = bH\) then \(a^2H = b^2H\).