Various problems relating to groups (an evolving list, check the date)

Most of the following may be found in many places – the references given are most certainly not to the first appearance, but they may help as pointers to preceding discussions of related topics.


1. [1, p.33, lemma 2.3] Suppose $G$ is a finite group and $H \subseteq G$ is a subset that is closed under multiplication. Then $H$ is a subgroup of $G$. (Hint: For $a \in H$ argue why $a^{-1} \in \{a^k : k \geq 1\}$.)

2. [1, p.40, corollary] Suppose $H$ and $K$ are subgroups of a finite group $G$, and both $|H| \geq \sqrt{|G|}$ and $|K| \geq \sqrt{|G|}$. Then $H \cap K \neq \{e\}$.

3. [1, p.41, exercise 17] Suppose $a, b$ are elements of a group that satisfy the relations $a^5 = e$ and $aba^{-1} = b^2$. Determine the order of $b$.

4. [1, p.58, lemma 2.19] Suppose $G$ is a group, $Z(G)$ its center. Then the group $I(G)$ of inner automorphisms of $G$ is isomorphic to $G/Z(G)$.

5. [1, p.45, exercise 14] Give an example of a finite group $G$ with subgroups $H, K \leq G$ such that $H$ is a normal subgroup of $K$ and $K$ is a normal subgroup of $H$ but $H$ is not a normal subgroup of $G$. (A starred problem in [1], discussed as example 35.3 in [4].)

6. [1, p.45, exercise 12] Suppose that $H, N \leq G$ are normal subgroups of a group $G$ and $H \cap N = \{e\}$. Show that $\forall n \in N$, $\forall h \in H$, $nh = hn$.

7. [1, p.55, exercise 5] Suppose $G$ is a group and $C(G) = \{xyx^{-1}y^{-1} : x, y \in G\}$ is its commutator subgroup. Show that (i) $C(G) \leq G$, (ii) the factor group $G/C(G)$ is abelian, (iii) If $N \leq G$ and the factor group $G/N$ is abelian then $N \supseteq C(G)$, and (iv) every subgroup $H$ of $G$ that contains $C(G)$ is normal in $G$.

8. [1, p.45, exercise 7] Give an example of a nonabelian group $G$ all of whose subgroups are normal in $G$.

9. [1, p.64, exercise 10] Suppose $p > q$ are primes. Show that if $q \nmid (p - 1)$ then every group $G$ of order $|G| = pq$ is cyclic, else there exists a nonabelian group $G$ of order $|G| = pq$.

10. Suppose $n > 4$ and $G \leq S_n$ is the subgroup of $S_n$ generated by the permutations $(1\ 4)(2\ 3)$ and $(1\ 2\ 3; \ldots n)$. Show that if $n$ is even then $G \cong S_n$ and if $n$ is odd then $G \cong A_n$.

11. Find all group homomorphisms from the cyclic group $\mathbb{Z}_6$ into the group group $G = \{1, 2, 3, 4, 5, 6\}$ under multiplication mod 7. Identify all group isomorphisms.

12. Suppose $G$ is a finite group such that for every $n \in \mathbb{Z}^+$ the subset $\{x \in G : x^n = e\}$ has at most $n$ elements. Show that $G$ is cyclic. (Application to finite fields). Show that the multiplicative groups of every finite field is cyclic.