1. No major issues here – I had hoped for a little better performance overall after having prepared / discussed so many specific examples in class. Very few mentioned all 7 subgroups . . . A surprising number of elementary arithmetic errors in 1.c., e.g. multiplying 18 and 30. Treat it like taxes, double-check all your arithmetic!

2. This question addresses one of the most fundamental constructions of this class, everyone should now this argument inside out. The key issue is well-definedness (every time a function is defined on equivalence classes by using representatives) [This may come up one every future exam again!]

As originally typed \((ah) \cdot (bH) = (ab)H\) simply makes no sense – what is \(h\)? It was my type-oh, but it should be clear from the context that the \(h\) had to be an \(H\). I was surprised at how much ill-defined computations this caused. The best advice is: “Do not use any symbol (variable) that has not been quantified, defined, or is a dummy-variable.”

3. This was homework, and we worked it in class – explicitly addressing a common mistakes made by a student in her/his presentation. While for many students this may have been boring “busy-work”, I learnt a lot while grading the exam: All too many students started with the variables in the wrong set, and all too often a too casual (lack of precise) quantification led into trouble.

My best advice: In problems of this kind, do not cut any corners, meticulously start with e.g. “suppose \(g \in G\) and \(x \in \Phi^{-1}(N)\). Define \(y = \Phi(x) \in N' \subseteq G'\).” In general: “Do not use any symbol (variable) that has not been quantified, defined, or is a dummy-variable.”

4. I apologize for the type-oh: As stated \(g^2 = \mu\) and hence \(H = \langle \varrho, \mu \rangle\) has order 4. But still for almost all \(\alpha \in S_4\) one computes that \(\alpha g \varrho^{-1} \notin H\) etc. and neither \(H\) nor \(K\) is normal in \(S_4\). But now, \(HK\) is, as expected, NOT a subgroup of \(S_4\). The idea of the problem had originally been (before the type-oh) that although neither \(H\) nor \(K\) is normal in \(S_4\), their product \(HK\) has cardinality \(|H| \cdot |K| / |H \cap K| = |S_4|\) and hence \(HK\) actually is a subgroup! (In words, normality of at least one of \(H\) and \(K\) is a sufficient, but not a necessary condition for \(HK\) to be a subgroup.)

5. In this exercise it is important to keep careful track of the domains of the functions. My preference is to define (for a group \(G\) and its set of subgroups \(S = \{H \in 2^G : H \subseteq G\}\)) the functions \(\Phi : G \times G \mapsto G\) by \(\Phi : (g, x) \mapsto gxg^{-1}\) and \(\overline{\Phi} : G \times S \mapsto 2^G\) by \(\overline{\Phi} : (g, H) \mapsto \{gxg^{-1} : x \in H\}\). (Many use \(\iota_g\) for both \(\Phi(g, \cdot)\) and \(\overline{\Phi}(g, \cdot)\).) The first step is to carefully verify that if \(H \in S\) then \(\overline{\Phi}(H) \subseteq S\), i.e. verify the conditions for being a subgroup. Checking that \(\overline{\Phi}\) defines an action is uneventful. Most common were mistakes that confused e.g. \(H \in S\) with \(H \subseteq S\) or \(H \subseteq S\).

For part c. I mainly wanted to see the words conjugate and normal, but various answers were acceptable.

Not wanting to create any real problems that addressed transitivity and faithfulness of groups actions, this was an almost frivolously simple vocabulary check: Since \(\overline{\Phi}\) preserves the cardinality, the action cannot be transitive unless \(G = \{e\}\) is trivial. If \(G\) is abelian then for all \(g \in G\) and all \(H \in S\), \(\overline{\Phi}(g, H) = H\) and the action is not faithful if \(|G| > 1\). However, there are some cases where the action is faithful – the smallest nonabelian group giving such an example: The action of \(S_3\) by conjugation (as above) on the subset of \(S\) consisting of the 2-element subgroups of \(S_3\) is faithful, and hence the action of \(S_3\) on all of its subgroups is faithful, too.

(Aside, several students confused transitivity of an action with transitivity of a relation.)

6. Think of these questions as similar to what we started with, e.g. the “left group axioms are equivalent to the group axioms”; maybe more like curiosities for those working in areas far away, but algebraists should be able to do this on the spot. I saw lots of circular arguments. The most common circular argument included \(0 \cdot x = 0, (x - x) \cdot x = 0, x^2 - x^2 = 0\) – often without careful quantification (what is \(x\)?) and without making clear which statement implies which (unclear logical order). But even with these, there is no hope to make this one work. Why?