1. No excuse here, this is such a prominent problem about the notion of a group that everyone must be able to work any of several versions at any time. (A similar item from this section are the cancellation laws... be careful with these!) Aside: The definition of identity should not be an existence statement (“\(\exists e \in G\) such that...”), but rather just give a name if there is such an element. Regarding mistakes, quantifiers are a key issue - note that in “(\(\exists e \in G\) \(ex = x\) for all \(x \in G\)” and that in “(\(\exists x' \in G\) \(x'x = e\) for all \(x \in G\)” the misplaced “for all \(x \in G\)” actually is to be understood [7] as to mean different things in each statement. Much better: place it correctly. A careful reading of the question would start with a set \(\{e, \varepsilon, \ldots\} \subseteq G\) of left identities and define left inverses for \(x \in G\) as elements \(x', x'' \in G\) such that \(x'x = e\) and \(x''x = \varepsilon\), etc. I.e. until left identities are shown to be unique, any def of left inverses should reflect this. Moreover do not write “the” until you have shown there is one (and only one). Till then use “a”.

2. MAT 444 introduces a very large number of technical terms – MANY definitions to know. I prefer paraphrasing of definitions (demonstrates understanding of the concept). Without precise knowledge of the definitions there is no hope to get theorems and proofs right. The division algorithm is MAT 300 material and played a very prominent role in our study of generators and subgroups of cyclic groups. No excuse here. Critical are the quantifiers, in the correct order and the requirement that the remainder be smaller than the divisor. Icing on the cake are uniqueness and the requirement that the divisor be a positive integer. “Finding the set of all ...” always has more than one part: find candidates, show that they do the job, and show that these are indeed all solutions. A common mistake is to assume that a set \(\{a, a^2, \ldots, a^n\}\) has \(n\) elements without careful justification why \(i \neq j \Rightarrow a^i \neq a^j\).

3. We are several weeks past section 6 – by now please use the compact criterion for subgroups in terms of \(xy^{-1}\). When showing that \(\Phi[G] \leq G\), please start with \(x, y \in \Phi(G)\), not with \(x, y \in G\). If really well written the second way may be acceptable – but why run the risk of writing bad math that just bad looks? This problem was still about simple maps between groups. In general, taking powers is not a homomorphism – from now on we shall rarely look at such maps! As for counterexamples, things go wrong when the group is not abelian – so start with the most simple nonabelian group. Almost any element \(a \in S_3\) and \(m = 2, 3\) will do the job, nothing fancy needed.

4. There are many ways to show that every group of order 6 has at least two elements of order 3, and I enjoyed seeing various arguments employed. When using Cauchy’s theorem be precise – it first of all is about the order of subgroups (and only indirectly about the order of group elements). I like best starting with two distinct non-identity elements \(a, b \in G\) of order 2. Then \(a \neq ab \neq b\) and \(ab \neq e\) since \(a^{-1} = a \neq b\). But now \((ab)^2 \neq e\), and hence \(|ab| > 2\), since otherwise \(\{e, a, b, ab\} \leq G\) which violates Cauchy’s theorem. Generalization of this will be addressed in the sections on the Sylow theorems, especially Cauchy’s theorem (36.3 in our book).

5. Here many students wrote much too much. I would like to see quick references to prior work, e.g. \(\{g \in G : ghg^{-1} = h\} = \cap_{a \in H}\{g \in G : gag^{-1} = a\}\), and intersections of subgroups are subgroups. We soon will consider many sets defined in ways similar to this, and it is of paramount importance to meticulously, correctly read “set notation”. In all examples here, the main (dummy) variable was \(g \in G\). Any solution attempt that started with \(a, b \in H\) is a sign of a major misunderstanding, misreading set-notation. I received many surprisingly wrong guesses of which subgroups are necessarily abelian. Most of these are easily laid to rest by considering \(a = e, H = \{e\}\) or \(H = G\) and \(G\) nonabelian. While these subgroups will play prominent roles in sections to come, in this test they were designed to test old material only, at the level of exercises 5/51 and 5/52. (The latter included the bonus question!)

6. Simple computations – here it was quite evident that some students had played quite a bit with permutation computations (like 9/30), while others seemed to be novices at this (presumably having avoided getting their hands any dirtier than absolutely required). Learn from this — play a lot! Factor group computations, and later group classifications à la Sylow are wonderful playgrounds, but you will need to take your own initiative, and go beyond the minimum assignments. Surprisingly many responses left out the factorial in the order of the symmetric group \(S_n\). Think about how large 12! is – this is a biggie (of a misjudgment)

7. This was very new material – I expected quite a few mistakes here, it was worth fewer points and graded more leniently (as far as it regarded intuition and judgment of cosets). In the next 2 weeks we will use lots of notation that is invites similar mistakes ... use this test problem as a reminder to be super careful when manipulating extremely compact and convenient symbols - but which often have very different rules for manipulation than just numbers. The majority of wrong answers are easily traced back to lack of well-definedness (e.g. defining a map \(\Phi(aH) = \text{expression using } a\), and too casual use of symbols. For all but the most advanced students I most strongly recommend:

- **Never use a symbol that has not been defined** (as in “Let \(x = \text{expression using previously defined variables}.\)” or quantified (e.g. “\(\forall x\)” with proper order of all quantifiers, and justification for every existential statement) **unless it is a dummy variable** (e.g. in a sum or in an integral).
- **Always connect any statements (equations, membership, . . . ) by words (“because”, “therefore”, or implication symbols “\(\Rightarrow\), “\(\rightarrow\), “\(\Leftrightarrow\)”).