1. a. Suppose $E \subseteq X$ and $F \subseteq Y$ are closed, i.e. $(X \setminus E) \subseteq X$ and $(Y \setminus F) \subseteq Y$ are open. Then $X \times Y \setminus (E \times F) = (X \setminus E) \times Y \cup X \times (Y \setminus F) \subseteq X \times Y$ open, and hence $E \times F \subseteq X \times Y$ is closed. b. Suppose $X$ is Hausdorff and $\Delta = \{(x, x) : x \in X\}$. Suppose $(x, y) \in X \times Y \setminus \Delta$, i.e. $x \neq y \in X$. Then there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$. Hence $(x, y) \in U \times V$ which is open in $X \times Y$, and $(U \times V) \cap \Delta = \emptyset$ since $U \cap V = \emptyset$, proving that $\Delta$ is closed.

2. a. Clearly the intervals $[a, b)$ with $a < b$ rational cover $\mathbb{R}$. Now suppose $a, b, c, d \in \mathbb{Q}$, $a < b$, $c < d$ and $z \in [a, b) \cap [c, d)$. Let $e = \max\{a, c\}$ and $f = \min\{b, d\}$. Clearly, $e, f \in \mathbb{Q}$ and hence $[e, f) \in B_{\mathbb{Q}}$. Since $z \in [e, f) \subseteq [a, b) \cap [c, d)$, this shows that $B_{\mathbb{Q}}$ is a basis for a $T_{\mathbb{Q}}$ on $\mathbb{R}$. b. We argue that the topology $T_{\mathbb{Q}}$ is strictly finer than $T_{\mathbb{R}}$, and hence the identity $i : (\mathbb{R}, T_{\mathbb{R}}) \to (\mathbb{R}, T_{\mathbb{Q}})$, is not continuous. Fix any $z \in \mathbb{R} \setminus \mathbb{Q}$ irrational and consider the interval $I = [z, \infty) \subseteq T_{\mathbb{Q}}$. If $I$ was open in $T_{\mathbb{Q}}$, then there would exist a basic open set $[p, q) \subseteq B_{\mathbb{R}}$ such that $z \in [p, q) \subseteq [z, \infty)$. But this is impossible since it implies $z < p \leq z$, i.e. $p = z \in \mathbb{Q} \cap (\mathbb{Q} \setminus \emptyset) = \emptyset$.

3. a. The closure of $A = \{(x, 1) : x \in \mathbb{Q} \}$ and $0 < x < 1 \}$ in $I_0^2$ is $\overline{A} = [0, 1) \times \{0, 1\} \times \{0\}$. The closure of $B = \{(x, \frac{1}{2}) : x \in \mathbb{Q} \}$ and $0 < x < 1 \}$ in $I_0^2$ is $\overline{B} = B \cup [0, 1) \times \{0\} \times \{0\}$. This follows from the observations that for every open subset $U \subseteq I_0^2$ that contains a point $(x, 0) \in [0, 1) \times \{0\}$, there exists $y > x$ such that $(x, y) \times [0, 1] \subseteq U$ and hence $U \cap A \neq \emptyset \neq U \cap B$. Argue analogously for points $(x, 0) \in [0, 1) \times \{1\}$. For $(0,0)$ and $(1,1)$, their open neighborhoods $[(0,0), (0,1)]$ and $[(1,0), (1,1)]$ intersect neither $A$ nor $B$. Finally for any $(x,y) \in [0,1) \times [0,1)$ its open neighborhood $\{x\} \times (0,1)$ does not intersect $A$ and intersects $B$ only if $(x,y) \in B$. Bonus. The sequences $(1 - \frac{1}{n^2})_{n=1}^\infty \subseteq A$ and $(1 - \frac{1}{n^2})_{n=1}^\infty \subseteq B$. The other direction follows by reversing the above arguments. b. Suppose $U = \prod_{\beta \in \Lambda} B_\beta$ is a basic open set in $Y$, i.e. there exists a finite set $I \subseteq \Lambda$ such that for all $\beta \in \Lambda \setminus I$, $U_\beta = \beta$ and hence $\beta^{-1}(U_\beta) = X$. Hence $f^{-1}(U) = \prod_{\beta \in \Lambda} \beta^{-1}(U_\beta)$ which by hypothesis is a finite intersection of open sets in $X$, and hence is open.

5. Suppose $X$ and $Y$ are metric spaces, $f : X \to Y$ is continuous, and $a \in X$ and $\varepsilon > 0$ are arbitrary but fixed. Then $B_Y(f(a), \varepsilon) = \{y \in Y : d_Y (y, f(a)) < \varepsilon \} \subseteq Y$ is open, and hence $f^{-1}(B_Y(f(a), \varepsilon)) \subseteq X$ is open, and clearly $a \in f^{-1}(B_Y(f(a), \varepsilon))$. Hence there exists $\delta > 0$ such that $B_X(a, \delta) \subseteq f^{-1}(B_Y(f(a), \varepsilon))$, or in other words, for every $x \in X$, if $d_X(x, a) < \delta$, then $d_Y(f(x), f(a)) < \varepsilon$.

6. Let $E \subseteq \mathbb{R}^\omega$ be the subset of all real valued sequences that are eventually zero. a. Suppose that $x = (x_n)_{n=1}^\infty \in \mathbb{R}^\omega$ and $U = \prod_{n=1}^\infty U_n$ is a basic open set containing $x$ in the product topology. Then there exists an infinite subset $I \subseteq \mathbb{Z}^+$ such that for every $n \in \mathbb{Z}^+ \setminus I$, $U_n = \mathbb{R}$. Let $N = \max I$ and define $y \in \mathbb{R}^\omega$ by $y_k = x_k$ if $k \leq N$ and $y_k = 0$ else. Then $y \in U \cap E$, showing that $x \in E$ and hence $\overline{E} = \mathbb{R}^\omega$. b. Suppose that $x = (x_n)_{n=1}^\infty \in \mathbb{R}^\omega \setminus E$. Then there exists an infinite subset $I \subseteq \mathbb{Z}^+$ such that w.l.o.g. for every $n \in I$, $x_n > 0$. Define the sequence $U_n = (0, \infty)$ if $n \in I$ and $U_n = X_n$ if $n \in \mathbb{Z}^+ \setminus I$. Then $U = \prod_{n=1}^\infty U_n$ is an open set in the box topology containing $x$ that has empty intersection with $E$. Hence $E$ is closed in the box topology. c. Suppose $x = (x_n)_{n=1}^\infty \in \mathbb{R}^\omega$ converges to zero, and $U \subseteq \mathbb{R}^\omega$ contains $x$ and is open in the uniform topology. Then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) = \{y \in \mathbb{R}^\omega : \sup_k |y_k - x_k| < \varepsilon \} \subseteq U$. Since $x$ converges to zero, there exists $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$, if $n > N$ then $x_n \in (-\varepsilon, \varepsilon)$. Define $y \in \mathbb{R}^\omega$ by $y_k = x_k$ if $k < N$ and $y_k = 0$ else. Then $y \in U \cap E$, showing that the closure of $E$ in the uniform topology contains all sequences converging to zero. Conversely, suppose that $x = (x_n)_{n=1}^\infty \in \mathbb{R}^\omega$ does not converge to 0, i.e. there exists $\varepsilon > 0$ and an infinite set $I \subseteq \mathbb{Z}^+$ such that for all $n \in I$, $|x_n| > \varepsilon$. Then $U = B(x, \varepsilon) = \{y \in \mathbb{R}^\omega : \sup_k |y_k - x_k| < \varepsilon \}$ is an open neighborhood of $x$ in the uniform topology that does not intersect $E$, and $x$ is not in the closure of $E$. 

MAT 410 Intro to General Topology
March 6, 2007
Test 1, sample solutions