Work as many problems as you can.

100 points constitute a perfect score.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

1. Suppose $X$ and $Y$ are topological spaces and $E \subseteq X$, $F \subseteq Y$ are closed subsets.
   a. Prove that $E \times F \subseteq X \times Y$ is closed (or give a counterexample, if not true).
   b. Suppose $X = Y$ and $X$ is Hausdorff. Prove that $\Delta = \{(x, x) : x \in X\}$ is closed.

2. On $\mathbb{R}$ consider the collection of intervals $B_{qt} = \{[a, b) : a < b \text{ and } a, b \in \mathbb{Q}\}$.
   a. Verify that $B_{qt}$ is a basis for a topology $T_{qt}$ on $\mathbb{R}$.
   b. Let $T_{l}$ denote the lower limit topology on $\mathbb{R}$. Decide whether the function $t : (\mathbb{R}, T_{qt}) \mapsto (\mathbb{R}, T_{l})$, defined by $t(x) = x$ is continuous. Justify your answer.

3. Consider the ordered square $I_o = [0, 1] \times [0, 1]$ (in the dictionary topology), and the subspaces $A = \{(x, 1) : x \in \mathbb{Q} \text{ and } 0 < x < 1\} \subseteq I_o$ and $B = \{(x, \frac{1}{2}) : x \in \mathbb{Q} \text{ and } 0 < x < 1\} \subseteq I_o$.
   a. Find the closures of $A$ and $B$ in $I_o$. Briefly justify your answers.
   **Bonus.** Construct, if possible, sequences $(a_n)_{n=1}^{\infty} \subseteq A$ and $(b_n)_{n=1}^{\infty} \subseteq B$ that converge to $(1,0) \in I_o$. Explain why you are correct, or prove that no such sequences exist.
   b. Prove or disprove: The function $f : A \mapsto B$, defined by $f(x, 1) = (x, \frac{1}{2})$ is continuous.

4. Suppose $X$ and $Y_\beta$, $\beta \in \Lambda$, are topological spaces, $Y = \prod_{\beta \in \Lambda} Y_\beta$ has the product topology, $f : X \mapsto Y$, and for each $\beta \in \Lambda$, $B_\beta \subseteq Y_\beta$ is nonempty , $\pi_\beta : Y \mapsto Y_\beta$ is the projection onto $Y_\beta$ and $f_\beta = \pi_\beta \circ f$.
   a. Prove or provide a counterexample: $f^{-1}(\bigcap_{\beta \in \Lambda} B_\beta) = \bigcap_{\beta \in \Lambda} f_\beta^{-1}(B_\beta)$
   b. Prove or provide a counterexample: If each $f_\beta$ is continuous, then $f$ is continuous.

5. Suppose $X$ and $Y$ are metric spaces and $f : X \mapsto Y$. Prove:
   If $f$ is continuous according to the open-set definition in general topological spaces then $f$ satisfies the $\varepsilon$-$\delta$-characterization of continuity in metric spaces.

6. Consider the subset $E \subseteq \mathbb{R}^\omega$ of all real valued sequences that are eventually zero.
   a. Find the closure of $E$ as a subspace of $\mathbb{R}^\omega$ in the product topology.
   b. Find the closure of $E$ as a subspace of $\mathbb{R}^\omega$ in the box topology.
   c. Find the closure of $E$ as a subspace of $\mathbb{R}^\omega$ in the uniform topology.
   **Briefly** justify your answers.