Lemma. Suppose \( d \) is a metric on a space \( X \). If \( f: [0, \infty) \to [0, \infty) \) is strictly increasing, concave, and \( f(0) = 0 \), then \( d' = f \circ d \) is also a metric on \( X \), and it generates the same topology as \( d \).

Proof. Positive definiteness and symmetry of \( d' \) are clear.

Now suppose that \( x, y, z \in X \) are arbitrary, but fixed. Since \( d \) is a metric we have

\[
d(x, z) \leq d(x, y) + d(y, z)
\]

Using the monotonicity of \( f \) it follows that

\[
d'(x, z) = f(d(x, z)) \leq f(d(x, y) + d(y, z))
\]

The concavity of \( f \), together with \( f(0) = 0 \), implies that for all \( a \geq 0 \) and all \( t > 0 \),

\[
\frac{f(a + t) - f(a)}{t} \leq \frac{f(t) - f(0)}{t - 0}
\]

and hence \( f(a + t) - f(a) \leq f(t) \) or \( f(a + t) \leq f(a) + f(t) \)

Applying this to the previous inequality with \( a = d(x, y) \) and \( t = d(y, z) \) yields

\[
d'(x, z) \leq d(d(x, y) + d(y, z)) \leq d(d(x, y)) + d(d(y, z)) = d'(x, y) + d'(y, z).
\]

Since \( x, y, z \in X \) were arbitrary, it follows that \( d' \) satisfies the triangle inequality and is a metric on \( M \). It generates the same topology as \( d' \) since for, e.g., all \( 0 \leq t \leq 1/2 \), \( t/2 \leq f(t) \leq t \).

Theorem. Suppose that \((X_k, d_k), k \in \mathbb{Z}^+\) is a countable collection of metric spaces. Then the product topology on \( X = \prod_{k=1}^{\infty} X_k \) is generated by the metric \( d: X \times X \to [0, \infty) \) defined by

\[
d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)},
\]

Proof. Applying the lemma to \( f: [0, \infty) \to [0, \infty) \) defined by \( f(t) = \frac{t}{1+t} \), shows that for every \( k \in \mathbb{Z}^+ \),

\[
2^{-k}(f \circ d_k)
\]

defines a metric on \( X_k \) that generates the same topology as \( d_k \). It readily follows that \( d \) is a metric on the product \( X \).

Let \( T \) denote the product topology on \( X \) and \( T_d \) denote the metric topology on \( X \) generated by \( d \).

We show that \( T_d \supseteq T \) and \( T \supseteq T_d \).

Suppose \( z \in X \) and \( U = \prod_{k=1}^{\infty} U_k \) is a basic open set in the product topology \( T \) on \( X \) that contains \( z \).

Then there exists \( N \in \mathbb{Z}^+ \) such that for all \( k \leq N \), \( U_k = X_k \).

For each \( k \leq N \) there exists \( \varepsilon_k > 0 \) such that the open ball \( B_k = \{ w \in X_k : d_k(w, z_k) < \varepsilon_k \} \subseteq U_k \).

Define \( \varepsilon = \min \{2^{-k} f(\varepsilon_k) : k \leq N \} \). Since the set is finite, \( \varepsilon > 0 \).

To verify that the open ball \( B(z, \varepsilon) \) is contained in \( U \), suppose \( y \in B(z, \varepsilon) \), i.e., \( d(z, y) < \varepsilon \). Since the infinite series is less than \( \varepsilon \), certainly each summand is less than \( \varepsilon \), i.e., for every \( k \in \mathbb{Z}^+ \), and in particular, for every \( k \leq N \), \( 2^{-k}(f \circ d_k)(y_k, z_k) < \varepsilon \), or equivalently \( d_k(y_k, z_k) < f^{-1}(2^{k}\varepsilon) \leq f^{-1}(2^{k}2^{-k}f(\varepsilon_k)) = \varepsilon_k \).

Consequently for every \( k \in \mathbb{Z}^+ \), \( y_k \in B_k \subseteq U_k \), and therefore \( B \subseteq U \) and \( T_d \supseteq T \).

Conversely suppose that \( z \in X \), \( \varepsilon > 0 \), and \( B(z, \varepsilon) = \{ y \in X : d(z, y) < \varepsilon \} \) is a basic open set in \( T - d \).

Choose \( N \in \mathbb{Z}^+ \) such that \( 2^{-N} < \frac{\varepsilon}{2} \). For \( k \leq N \) define \( U_k = \{ w \in X_k : d_k(y, z) < \frac{\varepsilon}{2N} \} \). For \( k > N \) define \( U_k = X_k \). Then \( U = \prod_{k=1}^{\infty} U_k \) is a basic open set in the product topology \( T \) on \( X \).

To verify that \( U \subseteq B(z, \varepsilon) \), suppose \( y \in U \). We show that \( y \in B \), hence \( U \subseteq B(z, \varepsilon) \), i.e. \( T \supseteq T_d \).

\[
d(z, y) = \sum_{k=1}^{N} 2^{-k} \frac{d_k(z_k, y_k)}{1 + d_k(z_k, y_k)} + \sum_{k=N+1}^{\infty} 2^{-k} \frac{d_k(z_k, y_k)}{1 + d_k(z_k, y_k)}
\]

\[
\leq \sum_{k=1}^{N} d_k(z_k, y_k) + \sum_{k=N+1}^{\infty} 2^{-k} \leq \left( \sum_{k=1}^{N} \frac{\varepsilon}{2N} \right) + 2^{-N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]