1. a. State the least upper bound axiom. Include definitions of upper bound and least upper bound.
   b. State the definitions of open and closed (subsets of a metric space).
   Briefly summarize the relationship between these two concepts.
   c. Briefly summarize the relationship between convergent sequences and Cauchy sequences.
   d. State the definition of compact. State a theorem that characterizes precisely which subsets of \( \mathbb{R} \) are compact. State a major theorem that requires compactness and continuity.
   Bonus. Give counterexamples that show that the theorem is not true without the assumptions of compactness and continuity.

2. In each of the following either prove the statement or give a counterexample.
   a. If a function \( f : \mathbb{R} \mapsto \mathbb{R} \) has a limit at \( a \in \mathbb{R} \), then the limit is unique.
   b. A countable, infinite set \( S = \{ a_n : n \in \mathbb{Z}^+ \} \subseteq \mathbb{R} \) cannot have exactly two limit points.
   Bonus. A set cannot have “more” limit points than it has elements.

3. a. Outline the key steps of an argument, working from the axioms of the real numbers, why there exists \( x \in \mathbb{R} \) such that \( x^2 = 2 \).
   Bonus: Give an alternative argument using the language of preimage, connected, continuous.
   b. Suppose that \( S \subseteq \mathbb{R} \) is a nonempty bounded set of real numbers that does not have a largest element. Show that there exists an increasing sequence \( (a_n)_{n=1}^{\infty} \) in \( S \) that converges to \( \sup S \).

4. a. State both forms of the Fundamental Theorem of Calculus.
   In each of the following either prove the statement or give a counterexample:
   b. If a function \( f : \mathbb{R} \mapsto \mathbb{R} \) is differentiable everywhere, then \( f \) is continuous everywhere.
   c. If a function \( f : \mathbb{R} \mapsto \mathbb{R} \) is differentiable everywhere, then \( f' \) is continuous everywhere.
   d. Suppose \( f : \mathbb{R} \mapsto \mathbb{R} \) is differentiable everywhere and \( f'(0) > 0 \).
   What can you conclude about \( f \)? Be very precise (and be very careful).

5. Suppose \( f : \mathbb{R} \mapsto \mathbb{R} \) is uniformly continuous and \( (a_n)_{n=1}^{\infty} \) is a Cauchy sequence of real numbers. Working from the definitions, prove that \( (f(a_n))_{n=1}^{\infty} \) is a Cauchy sequence.
   Bonus. Give an example of a continuous function \( f : S \mapsto \mathbb{R} \) (for some \( S \subseteq \mathbb{R} \)) and a Cauchy sequence \( (a_n)_{n=1}^{\infty} \) of real numbers such that \( (f(a_n))_{n=1}^{\infty} \) is a not Cauchy sequence.

6. Suppose that \( a, b \in \mathbb{R} \), \( a < b \), and \( f : [a, b] \mapsto [0, \infty) \). Working from the definition of the Riemann integral show that if \( f \) is unbounded then \( f \) is not integrable over \([a, b]\).

7. Suppose that \( a, b \in \mathbb{R} \), \( a < b \) and \( (f_n)_{n=1}^{\infty} \) is a sequence of continuous functions each defined on \([a, b]\) that converges uniformly to a function \( f \). Prove that \( f \) is continuous.
   Bonus. Prove that \( \int_a^b f = \lim_{n \to \infty} \int_a^b f_n \).
   Bonus. Give counterexamples that show that the conclusions need not hold if the convergence is not uniform, or if the interval \([a, b]\) is replaced by \( \mathbb{R} \).