MAT 371 Advanced Calculus October 2004

Uniform continuity: Explicit calculations for the example $\sqrt{x}$

**Proposition:** The function $f: \mathbb{R} \mapsto \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

**Discussion:** Eventually this will be an immediate consequence of general theorems that provide e.g. continuity of polynomial functions, continuity of inverse functions, and uniform continuity of continuous functions on compact sets. Still it is instructive to try to give *bare-handed* direct proofs using only precalculus and algebra. – But think about how much work one such example is, and use this as motivation for making a general theory, necessarily somewhat abstract – but which then handles all such examples.

The most interesting point is clearly $x = 0$ where the function is not differentiable. It is intuitively clear that for a given $\delta > 0$ the function $E: x \mapsto \sqrt{x + \delta} - \sqrt{x}$ attains its maximum $E_{\text{max}} = 2\sqrt{\tfrac{\delta}{2}}$ at $x = -\tfrac{\delta}{2}$. Hence for any given $\epsilon > 0$ the choice $\delta = \tfrac{1}{4}\epsilon^2$ will do the job. There are many ways to make this into a rigorous argument. One interesting fair-play requirement is to not utilize derivatives. There are many possible avenues – below are two examples. Alternatives are to use the homogeneity $d(f(y), f(x)) = x^{1/3}d(f(1+h), f(1))$ with $h = \tfrac{y}{x}$ for $x \neq 0$ together with e.g. Bernoulli’s inequality etc.

**Proof 1** (brute-force, bare-handed estimates)

Let $\epsilon > 0$ be given. Define $\delta = \tfrac{1}{4}\epsilon^3$. Suppose $x, y \in \mathbb{R}$ such that $d(x, y) < \delta$. W.l.o.g. assume $y > x$. Since $f$ is the inverse of the monotonically increasing function $f^{-1}(x) = x^3$, $f$ is also monotonically increasing, and hence $d(f(y), f(x)) < f(x + \delta) - f(x)$. Estimate

$$
(f(x + \delta) - f(x))^3 = \left((x + \delta)^{1/3} - x^{1/3}\right)^3
= (x + \delta) - 3x^{1/3}(x + \delta)^{2/3} + 3x^{2/3}(x + \delta)^{1/3} - x
= \delta - 3x^{1/3}(x + \delta)^{1/3} \left((x + \delta)^{1/3} - x^{1/3}\right)
< 0 \text{ only if } -\delta < x < 0 \text{ always } \geq 0
$$

$$
\leq \delta + 3 \cdot \max_{-\delta < x < 0} |x(x + \delta)|^{1/3} \cdot \left(\max_{-\delta < x < 0} |x + \delta|^{1/3} + \max_{-\delta < x < 0} |x|^{1/3}\right)
< \delta + 3 \cdot \left(\tfrac{1}{4}\delta^2\right)^{1/3} \cdot (\delta^{1/3} + \delta^{1/3}) \leq 7\delta = \epsilon^3.
$$

This shows that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$. ■

**Proof 2** (using concavity/convexity, but no derivatives)

It is easily shown by induction – no calculus required – that every power function $x \mapsto x^n$ for $n \in \mathbb{Z}^+$ and $x \geq 0$ is strictly increasing and convex. Hence for $x \geq 0$ and $n \in \mathbb{Z}^+$ each function $x \mapsto x^{1/n}$ is strictly increasing and concave. This implies that if $0 \leq x < y$ then

$$
\frac{f(y) - f(x)}{y - x} \leq \frac{f(y - x) - f(0)}{(y - x) - 0} \quad \text{and hence} \quad f(y) - f(x) \leq f(y - x).
$$

Alternatively, in the special case of the function $f(x) = x^{1/3}$ a direct algebraic argument is

$$
0 < 3((y - x)xy)^{1/3} = \left((y^{1/3} - x^{1/3})^3 - (y - x) = (f(y) - f(x))^3 - (f(y) - x)^3
$$

In the case that $xy < 0$ directly use concavity together with the symmetry $f(x) = -f(-x)$

$$
d(f(y), f(x)) = |f(y) - f(x)| = |f(y) + f(-x)| \leq 2|f\left(\frac{y-x}{2}\right)|
$$

Thus given $\epsilon > 0$ define $\delta = \frac{1}{4}\epsilon^3$. Suppose $x, y \in \mathbb{R}$ and $d(x, y) < \delta$. If $xy \geq 0$ then $d(f(y), f(x)) \leq f(d(y, x)) < \delta^{1/3}/\epsilon$. If $xy < 0$ then $d(f(y), f(x)) \leq 2f\left(\frac{1}{2}d(y, x)\right) < 2\left(\frac{1}{2}\delta\right)^{1/3} = \epsilon$. 