Both the theorem and the proof are wrong. It is your job to find the mistakes (more than one!), explain what is wrong, and then correct both the theorem and the proof.

The references are to the textbook’s Closer and Closer by C. Schumacher.

**Theorem:** Suppose $S$ is a subset of a metric space $(X,d)$, $z \in X \setminus S$ is a limit point of $S$, and $f:S \mapsto \mathbb{R}$ is continuous. If $f$ is bounded, then there exists a unique continuous extension $\overline{f}:S \cup \{z\} \mapsto \mathbb{R}$ of $f$. Moreover $\overline{f}$ is bounded by the same bounds as $f$.

**Proof:**

1. Suppose $S$ is a subset of a metric space $(X,d)$, $z \in X \setminus S$ is a limit point of $S$, and $f:S \mapsto \mathbb{R}$ is continuous and bounded.
2. Since $z$ is a limit point of $S$ there exists a sequence $(a_n)_{n=1}^{\infty} \subseteq S$ that converges to $z$.
3. Since $f$ is bounded on $S$, there exists $M \in \mathbb{R}$ such that $|f(a_n)| \leq M$ for all $n \in \mathbb{Z}^+$.
4. By theorem 3.4.9. this sequence has a converging subsequence $(f(a_{n_k}))_{k=1}^{\infty}$.
5. Let $L \in \mathbb{R}$ be the unique limit of this subsequence. By theorem 3.4.3 $-M \leq L \leq M$.
6. Define $\overline{f}:S \cup \{z\} \mapsto \mathbb{R}$ by $\overline{f}(x) = f(x)$ for $x \in S$ and $\overline{f}(z) = L$.
7. We will show that $\overline{f}$ is continuous. Let $\varepsilon > 0$ be given.
8. Since $(f(a_{n_k}))_{k=1}^{\infty}$ converges to $L$, there exists $N_1 \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$, if $k > N_1$ then $d(f(a_{n_k}), L) < \frac{\varepsilon}{2}$.
9. Since $a_{n_k} \in S$ and $f$ is continuous on $S$ there exists $\delta > 0$ such that if $x \in S$ and $d(x, a_{n_k}) < \delta$ then $d(f(x), f(a_{n_k})) < \frac{\varepsilon}{2}$.
10. Since $(a_n)_{n=1}^{\infty}$ converges to $z$, the subsequence $(a_{n_k})_{k=1}^{\infty}$ also converges to $z$.
11. Hence there exists $N_2 \in \mathbb{Z}^+$, w.l.o.g. $N_2 \geq N_1$, such that for all $k \in \mathbb{Z}^+$, if $k > N_2$ then $d(a_{n_k}, z) < \frac{\delta}{2}$.
12. Suppose $k > N_2$ and $x \in B_{\frac{\delta}{2}}(z)$, then also $x \in B_{\delta}(a_{n_k})$ because of the triangle inequality $d(x, a_{n_k}) \leq d(x, z) + d(z, a_{n_k}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$.
13. Thus if $k > N_2$ and $x \in B_{\delta}(z)$ then
\[
d(\overline{f}(x), \overline{f}(z)) = d(f(x), L) \leq d(f(x), f(a_{n_k})) + d(f(a_{n_k}), L) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Both the theorem and the proof are wrong. It is your job to find the mistakes (more than one!), explain what is wrong, and then correct both the theorem and the proof.

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**Proof:**

1. Suppose $S$ is a subset of a metric space $(X, d)$, $z \in X \setminus S$ is a limit point of $S$, and $f: S \mapsto \mathbb{R}$ is continuous and bounded.

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8. Since $(f(a_{n_k}))_{k=1}^{\infty}$ converges to $L$, there exists $N_1 \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$, if $k > N_1$ then $d(f(a_{n_k}), L) < \frac{\varepsilon}{2}$.

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10'. Hence there exists $\ell \in \mathbb{Z}^+$, w.l.o.g. $\ell \geq N_1$, such that $d(a_{n_{\ell}}, z) < \frac{\delta}{2}$.

11'. Since $a_{n_{\ell}} \in S$ and $f$ is continuous on $S$, $f$ is continuous at $a_{n_{\ell}}$.

   Thus there exists $\delta > 0$ such that if $x \in S$ and $d(x, a_{n_{\ell}}) < \delta$ then $d(f(x), f(a_{n_{\ell}})) < \frac{\varepsilon}{2}$.

12'. Suppose $x \in B_{\delta/2}(z)$, then also $x \in B_{\delta}(a_{n_{\ell}})$ because of the triangle inequality

   $d(x, a_{n_{\ell}}) \leq d(x, z) + d(z, a_{n_{\ell}}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$.

13'. Thus if $x \in B_{\delta}(z)$ then

   $d(\overline{f}(x), \overline{f}(z)) = d(f(x), L) \leq d(f(x), f(a_{n_{\ell}})) + d(f(a_{n_{\ell}}), L) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. 