The general outline of the construction of the real numbers should be known to every mathematics major, but working out all details takes quite some effort. The steps outlined vary between simple review and quite challenging – any student in the class is encouraged to work out as many steps as feasible . . .

There exist different classical approaches to constructing the real numbers, with Dedekind cuts likely being the most well-known. The outline given here has the advantage of being a model for analogous constructions at more advanced stages – e.g. polynomials or other nice functions take the place of the rationals, and generalized functions (alas measures, including such objects as the Dirac delta function) are constructed in an analogous way (a critical feature is to use a suitable notion of distance or norm that generalizes $\|x\|$ or $d(x,y) = |y - x|$.)

- Let $S$ denote the set of all sequences $a: \mathbb{Z}^+ \to \mathbb{Q}$ of rational numbers. As a set of functions with values in the field $\mathbb{Q}$, the set $S$ inherits natural additive and multiplicative structures defined by
  
  $$(a + b)_n \overset{\text{def}}{=} a_n + b_n \\
  (a \cdot b)_n \overset{\text{def}}{=} a_n \cdot b_n$$

  1. Check which of the field axioms are satisfied by $(S, +, \cdot)$. In particular, which sequences are the additive and multiplicative neutral elements? Which field axiom(s) is (are) not satisfied?

- Let $C \subseteq S$ be the subset of all Cauchy sequences. (For the sake of clear argumentation, consider only rational $\varepsilon > 0$, or only those $\varepsilon$ that are such that $1/\varepsilon \in \mathbb{Z}^+$.)

  2. Verify that the set $C$ is closed under addition and multiplication (you may simply refer to completed homework problems).

- Define a relation $\leq \subseteq C \times C$ by $a \leq b$ if $(\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall n > N) (a_n < b_n + \varepsilon)$.

  3.a. Give an example of $a, b \in C$ such that $a \leq b$ and $(\forall n \in \mathbb{Z}^+) (a_n > b_n)$.

  3.b. Verify that $\leq$ is a partial order on $C$.

- Define a relation $\sim \subseteq C \times C$ by $a \sim b$ if $(\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall n, m > N) (|a_n - b_m| < \varepsilon)$.

  4.a. Verify that $\sim$ is an equivalence relation.

  4.b. Give explicit examples of several Cauchy sequences that are related and some that are not related.

- Let $\mathbb{R} = C/\sim = \{[a] : a \in C\}$ where $[a] = \{b \in C : b \sim a\}$ is the equivalence class of $a \in C$.

  5.a. Review the construction of the set $\mathbb{Q}$ of rational numbers as the set of equivalence classes $(\mathbb{Z} \times \mathbb{Z}^+) /\sim$ under the equivalence relation $(p, q) \sim (r, s) \iff p \cdot s = q \cdot r$.

  5.b. Give explicit examples of several Cauchy sequences that belong to the same or to different equivalence classes.

- Verify that $\mathbb{R}$ inherits natural additive and multiplicative structures, i.e. if $x, y \in \mathbb{R}$ define $x + y$ and $x \cdot y$ by picking any representatives $a \in x$ and $b \in y$ (i.e. $x = [a]$ and $y = [b]$) and define $x + y = [a + b]$ and $x \cdot y = [a \cdot b]$. 
6.a. Show that $x + y$ is well-defined for $x, y \in \mathbb{R}$: I.e. if $x = [a] = [a']$ and $y = [b] = [b']$ for Cauchy sequences $a, a', b, b', \in C$, show that $[a + b] = [a' + b'].$

6.b. Show that $x \cdot y$ is well-defined for $x, y \in \mathbb{R}$.

6.c. Show that there exists a neutral element for addition in $\mathbb{R}$. Describe this element in detail – and give several examples of $a \in C$ such that $a \neq 0$.

6.d. Verify that $(\mathbb{R}, +, \cdot)$ satisfies the field axioms. Some of the trickiest parts deal with Cauchy sequences for which $a_n = 0$ for some $n$, and demonstrating the existence of multiplicative inverses for all $x \in \mathbb{R}$ that are different from the neutral element for addition.

- Define a relation $< \subseteq \mathbb{R} \times \mathbb{R}$ by $x < y \iff (x \neq y) \wedge ((\forall a \in x)(\forall b \in y)(\exists N)(\forall n > N)(a_n > b_n)).$

7.a. Verify that $<$ is well-defined (this takes quite a few steps!)

7.b. Recall the example from 3.a. and explain why this does not cause any problems here. Explain how $\leq$ on $C$ is related to $<$ on $\mathbb{R}$.

7.c. Show that $<$ defines an order relation on $\mathbb{R}$ that is compatible with its additive and multiplicative structures, i.e. $(\mathbb{R}, +, \cdot, <)$ satisfies the ordered field axioms.

- The 4-tuple $(\mathbb{R}, +, \cdot, <)$ is a representation of the real numbers:

8.a. Show that in addition to the ordered field axioms (see above), $(\mathbb{R}, +, \cdot, <)$ also satisfies the suprema-axiom, i.e. it has the least upper bound property.

- Up to natural identifications, the 4 tuple $(\mathbb{R}, +, \cdot, <)$ is the unique structure that satisfies the real number axioms:

9.a. Suppose that $\mathbb{R}'$ is a set endowed with two binary operations $\oplus$ and $\odot$ and a relation $\ll$ that satisfies the real number axioms. Show that there exists a bijective function $F: \mathbb{R}' \rightarrow \mathbb{R}$ that preserves order, sums, and products. (This takes quite a bit of work, and is typically only appreciated after studying quite a bit of higher math – here it is mainly included for the sake of completeness of the exposition. The first step is to identify copies of the natural numbers in each set $\mathbb{R}$ and $\mathbb{R}', \ldots$)

9.b. Explore a different representation that starts with nested interval sequences of rationals, i.e. replaces the set $C$ by the set $N$ consisting of sequences of pairs $(a_n, b_n)$ of rational numbers such that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for all $n$, and $(\forall \varepsilon > 0) (\exists N) (\forall n > N) (b_n - a_n < \varepsilon)$. (Basically repeat all the steps above with suitable modifications. Eventually, show that this construction gives rise to essentially the same real numbers as the construction worked out in detail before: Explicitly define the bijective function $F$ between these two representations.)

**Bonus:** The 4-tuple $(\mathbb{R}, +, \cdot, <)$ is a complete metric space.

10.a. Define a distance function ("metric") $d: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ by $d(x, y) = x - y$ if $x > y$, $d(x, y) = 0$ if $x = y$, and $d(x, y) = y - x$ if $x < y$. (Conveniently use the notation $|x| = d(x, 0)$.) Show that $d$ satisfies the axioms of a metric: $d(x, y) \geq 0$ for all $x, y$ and $d(x, y) = 0$ only if $x = y$, $d(x, y) = d(y, x)$ for all $x, y$, and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z$.

Call a sequence $x = (x_n)_{n \in \mathbb{Z}^+} \subseteq \mathbb{R}$ of real numbers (what is this kind of object in terms of equivalence classes, sequences, and rationals?) a Cauchy sequence if $(\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall n, m > N) (|x_n - x_m| < \varepsilon)$. ([with $\varepsilon$ rational, real, or such that $\frac{1}{\varepsilon} \in \mathbb{Z}^+]$)

10.c. Show that $(\mathbb{R}, +, \cdot, <)$ is complete, i.e. if $x = (x_n)_{n \in \mathbb{Z}^+} \subseteq \mathbb{R}$ is a Cauchy sequence then there exists $z \in \mathbb{R}$ such that $x_n \rightarrow z$. 