Consider the partial differential equation with boundary conditions defined by

(PDE) \[ u_{tt} - u_{xx} = 0 \] for \( 0 < x < 5\pi \) and \( 0 < t \),

(BC1) \[ u(0, t) = 0 \] for \( t \geq 0 \),

(BC2) \[ u(5\pi, t) = 0 \] for \( t \geq 0 \),

(BC3) \[ u(x, 0) = f(x) \] for \( 0 \leq x \leq 5\pi \), and

(BC4) \[ u_t(x, 0) = 0 \] for \( 0 \leq x \leq 5\pi \).

1. a. Can the function \( f(x) \) be expanded into a Fourier series?
   
   Can it be written as a Fourier integral?
   
   Can it be written as a cosine series?

   b. Sketch an “odd periodic extension” \( f_{\text{ext}}(x) \) of \( f(x) \) for \(-30 \leq x \leq 30\).

   c. Expand \( f_{\text{ext}}(x) \) as a Fourier series:
      
      (i) Find a general, reasonably simplified, formula for the Fourier coefficients.

      **Caution:** \( b_5 \) will need special consideration! Work this integral separately.

      (ii) Give 4-digit-accuracy decimal approx’s of the first five nonzero Fourier coefficients.

      (iii) Using (ii), write out the first five nonzero terms of the Fourier expansion.

   d. Use Parseval’s identity to find the relative mean-square error of this 5-term approximation.

2. a. Describe a physical phenomenon modeled by the (PDE). Explain, in terms of this physical scenario the reason for the plus sign in \( u_{tt} = +u_{xx} \) (i.e. as supposed to \( u_{tt} = -u_{xx} \)).

   b. Explain why this (PDE) is linear.

   Explain how this property is used in the method of separation of variables.

   c. Classify the (PDE) as hyperbolic, elliptic, parabolic, or none of these. Explain!

   d. Describe, in words, d’Alembert’s solution, and sketch the cross-sections \( u(x, t_i) \), \( 0 \leq x \leq 5\pi \) for several values of \( t_i \geq 0 \). Your “movie” should capture interesting features of the solution.

   e. Solve the (PDE) with (BC1-4) using separation of variables – explaining all major steps.

   (In particular, highlight the role of linearity, when and how you use which boundary conditions, how these determine the signs of any constants and the frequencies.) The solution should be an infinite series with reasonably simplified formulas for the Fourier coefficients.

   **You should spend at least as much effort on explaining how the method works as on manipulating formulas.** When using MAPLE, document only major steps.

   f. Write out the first five nonzero terms of your solution, and use these to
      
      (i) sketch the graph of (an approximation of) \( x \mapsto u(x, \pi) \),

      (ii) sketch the graph of (an approximation of) \( t \mapsto u(2\pi, t) \),

      (iii) calculate a 3-digit accuracy decimal approximation of \( u(4, 7) \).