1.a. Use separation of variables and Fourier expansions to solve the (PDE) $u_{tt} = u_{xx}$ in the domain $0 < x < \pi$ and $0 < t$, with boundary conditions (BC1,BC2) $u(0,t) = u(\pi,t) = 0$ for all $t \geq 0$, (BC3) $u_t(x,0) = 0$ for all $0 \leq x \leq \pi$, and (BC4) $u(x,0) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{4} \\ \frac{\pi}{2} - x & \text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \leq \pi \end{cases}$. In this exercise: Explanation may be kept very short. The emphasis is to demonstrate that you can calculate effectively. Simplify only as far as needed so that e.g. a finite Fourier approximation may be evaluated, plotted, or animated easily.

b. Write out explicitly a fifth order Fourier approximation of the solution $u(x,t)$ (with explicit numerical values for the Fourier coefficients).

c. Sketch the graph of $u(x,t_i)$ (as a function of $x$) for several values of $t_i$, e.g. $t_i = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$. You may use a Fourier approximation like in b. in place of the exact solution.

d. What physical phenomena / objects are modeled by this (PDE) and (BCs)? Is (BC4) a reasonable boundary condition (that is easily created in an experimental set-up)? (E.g. “how many fingers does it take” to produce these (BCs)? – Explain how!)

**Bonus:** Graphically demonstrate how the solution $u(x,t)$ can be considered the superposition of two traveling waves – i.e. sketch these two as functions of $x$ for several values of $t$.

2.a. Classify the (PDE) $u_t - u_{xx} = 0$ as hyperbolic, elliptic or parabolic.

b. Is this a linear partial differential equation? Why?

c. Briefly describe a physical phenomenon modeled by the (PDE).

Explain – in terms of your physical scenario, e.g. understandable by first-year calculus students – why the model uses $u_t - u_{xx} = 0$ and not $u_t + u_{xx} = 0$.

d. Continuing with this physical example, what do the boundary conditions (BC12) $u(0,t) = u(L,t) = 0$ and (BC12’) $u_x(0,t) = u_x(L,t) = 0$, respectively, mean in practical terms?

3. Consider the two-dimensional wave equation (PDE) $u_{tt} = u_{xx} + u_{yy}$ on the square $S$ given by $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ with boundary conditions (BC1,BC2) $u(x,0,t) = u(x,\pi,t) = 0$ for all $0 \leq x \leq \pi$, $0 \leq t$ and (BC3,BC4) $u(0,y,t) = u(\pi,y,t) = 0$ for all $0 \leq y \leq \pi$, $0 \leq t$.

a. Find all values of $(m,n,k)$ for which $u_{mnk}(x,y,t) = \sin mx \sin ny \cos kt$ is a solution of (PDE) together with the boundary conditions (BC1)-(BC4). Carefully explain how each of (PDE) and the (BCs) places constraints on which of $m$, $n$, and $k$, respectively.

b. Verify directly that (or use theory to explain why) $w = \cos \sqrt{5} t (2 \sin x \sin 2y - 3 \sin 2x \sin y)$ is a solution of (PDE) with (BC1) - (BC4).

c. Calculate and sketch the nodal lines of this solution $w(x,y,t)$ from part b., i.e. find all points $(\bar{x}, \bar{y})$ in the square $S$ such that $w(\bar{x}, \bar{y}, t) = 0$ for all $t \geq 0$.

*Hint:* The identity sin $2\alpha = 2\sin \alpha \cos \alpha$ might be helpful.