1.a Look for solutions of (PDE) together with (BC1-3) that are of the form $u(t,x) = F(x)G(t)$. Substituting into (PDE) yields, after rearranging, $F''(x)/F(x) = G''(t)/G(t)$. Since the left side does not depend on $t$, neither can the right. Since the right side does not depend on $x$ either, each side is a constant, call it $-n^2$ (for some – at this time – complex number $n$). The general solution of $F''(x) + n^2 F(x) = 0$ is $F(x) = c_1 \cos(nx) + c_2 \sin(nx)$ (any constants $c_1, c_2$). (BC1) forces $c_1 = 0$ and (BC2) forces that $n$ is an integer. The general solution of $G''(t) + n^2 G(t) = 0$ is $G(t) = C_1 \cos(nt) + C_2 \sin(nt)$ (any constants $C_1, C_2$). (BC3) forces $C_2 = 0$. Hence any solution of (PDE) with (BC1-3) that is in product form is a multiple of $u_n(t,x) = \sin nx \cos nt$.

To also satisfy (BC4) consider infinite linear combinations $u(t,x) = \sum_{n=1}^{\infty} b_n \sin nx \cos nt$. Evaluating at $t = 0$ and equating $u(x,0)$ according to (BC4) forces that $b_n = \frac{2}{\pi} \left( \int_0^{\pi/4} x \sin nx \, dx + \int_0^{\pi/4} \left( \frac{\pi}{4} - x \right) \sin nx \, dx \right) = \frac{2}{\pi n^2} \cdot \left( \sin \left( \frac{n\pi}{4} \right) - 2 \sin \left( \frac{n\pi}{4} \right) \right)$.

b. Evaluating the first five Fourier coefficients numerically yields the approximate solution $u(x,t) \approx \frac{1}{2} \left( 2(\sqrt{2}-1) \sin x \cos t + \sin 2x \cos 2t + \frac{2}{5}(1+\sqrt{2}) \sin 3x \cos 3t - \frac{2}{\sqrt{3}}(1+\sqrt{2}) \sin 5x \cos 5t \right)$.

c. For plots see e.g. the MAPLE worksheet (e.g. if $u$ is defined as above, use e.g. `plot(subs(t=Pi/4,u),x=0..Pi);`).

d. The (PDE) together w/ (BC1-2) model e.g. a vibrating string (of length $\pi$) with fixed endpoints. (BC3) says that the initial velocity is zero, and (BC4) describes the initial deflection – experimentally, this is easily obtained by using one finger to pressing down the center of the string fixed while plucking the string at one quarter of its lengths with a second finger.

Bonus. Call the MATLAB script `wave1.m` – preprogrammed with example 4.

2. a. Parabolic.

b. Linear: Write as $L(u) = 0$ with $L(u) = u_t - u_{xx}$.

Linearity means that $L(cu_1 + u_2) = cL(u_1) + L(u_2)$ for any functions $u_1, u_2$ and any constant $c$.

The superposition principle applies, i.e. if $u_1$ and $u_2$ are solutions, then so is $cu_1 + u_2$ for any constant $c$.

c. The (PDE) models e.g. temperature in a rod (wire) (or spread of an infection, of a gene, diffusion of a chemical etc. in one dimension). The sign means that at points where (for fixed $t$) $u$ is concave up (down) as a function of $x$, the temperature increases (decreases) – roughly speaking this means that the temperature at any point tends to the average of the neighboring points.

The opposite sign would instead aggravate the differences between neighboring points.

d. The first (BCs) mean that the temperature at the endpoints is kept fixed (e.g. dip into ice water). The second (BCs) model insulated endpoints, i.e. the endpoints have in first approximation the same temperature as nearby points (towards the center of the rod).

3.a Substituting $u_{mnk}(x, y, t) = \sin mx \sin ny \cos kt$ yields $(m^2 + n^2 - k^2) \cdot u_{mnk}(x, y, t) = 0$ for all $(x, y, t)$. Thus either $u_{mnk} = 0$, or $k^2 = m^2 + n^2$.

The boundary conditions (BC2) and (BC4) yield $0 = \sin m\pi \sin ny \cos kt$ for all $(y,t)$, and $0 = \sin mx \sin n\pi \cos kt$ for all $(x,t)$. Since $\cos kt \neq 0$, and similarly $\sin ny \neq 0$, $\sin mx \neq 0$, this forces that $m\pi$ and $n\pi$ are integer multiples of $\pi$, i.e. $m$ and $n$ must be integers.

b. Either substitute $w$ directly into (PDE) and (BC1-4) and simplify, or observe that (PDE) is a linear homogeneous (PDE) with zero boundary conditions (BC1-4). By part a. each of the two terms in $w(x, y, t)$ is a solution, and by linearity (superposition principle) so it $w(x, y, t)$.

c. Along the nodal curves $w(x, y, t) = 0$ for all $t$. Since $\cos kt \neq 0$, this means that $\sin x \sin y \sin 2(\cos y - 3 \cos x) = 0$ (using the suggested trigonometric identity). This is satisfied along the edges $x = 0$, $x = \pi$, $y = 0$, $y = \pi$ and along the curve $y = \arccos(\frac{3}{2} \cos(2x))$. [Compare textbook exa.11.8/1 and MAPLE worksheet wave2.mws.]