MAT 362 Advanced Math for Scientists and Engineers

December 12, 2000 Final Exam

Explain what you are doing. Computer printouts without explanations and formulas scattered over a page without clear logical order will be ignored (zero credit!) It is YOUR responsibility to demonstrate that you have mastered the material of this class. Check ALL results with available computer software!

1.a. Expand the function \( f \) given below in a Fourier sine-series

\[
f(t) = \begin{cases} 
  t & \text{if } 0 \leq t \leq \frac{\pi}{3} \\
  \frac{\pi}{3} & \text{if } \frac{\pi}{3} \leq t \leq \frac{2\pi}{3} \\
  \pi - t & \text{if } \frac{2\pi}{3} \leq t \leq \pi
\end{cases}
\]

b. Sketch the graph of \( f \) and overlay the graphs of the Fourier approximations \( f_1, f_3 \) and \( f_5 \) where

\[
f_n(t) = \sum_{k=1}^{n} b_k \sin kt
\]

c. Find \( N \) such that \( b_N \) is the fifth nonzero Fourier coefficient. Write out explicitly the approximation \( f_N \) (with 4-digit accuracy decimal approximations of the Fourier coefficients \( b_k \)).

d. Calculate \( \int_{0}^{\pi} |f(t)|^2 \, dt \) and \( \sum_{k=1}^{N} |b_k|^2 \). Use these, together with Bessel’s inequality, to estimate the total square-error

\[
E_N = \int_{0}^{\pi} |f(t) - f_N(t)|^2 \, dt = \left( \frac{2}{\pi} \right) \sum_{k=N+1}^{\infty} b_k^2
\]

in the approximation \( f_N \) of \( f \).

2.a. Use separation of variables and Fourier expansions to solve the (PDE) \( u_{xx} + u_{yy} = 0 \) in the domain \( 0 < x < \pi \) and \( 0 < y < \pi \), with boundary conditions (BC1,BC2) \( u(0,y) = u(\pi,y) = 0 \) for all \( 0 \leq y \leq \pi \), (BC3,BC4) \( u(x,0) = 0, u(x,\pi) = f(x) \) for all \( 0 \leq x \leq \pi \), with \( f \) as in 1.

**Explain how** each of (PDE) and the (BCs) is used in which step. In particular, point out where the eigenvalues come from, and why some constants must be zero, negative, or positive.

b. Write out the first five nonzero terms in the Fourier approximation of the solution \( u(x,y) \).

c. Find an approximate numerical value of \( u\left(\frac{\pi}{2},\frac{\pi}{2}\right) \).

**Bonus:** Plot the graph, a contour diagram, or suitable cross-sections of the solution.

Document your work, label and explain what you are showing.

d. What physical phenomena/objects may be modeled by this partial differential equation?

Is (BC4) a reasonable boundary condition (that is easily created in an experimental set-up)?

e. Solve the (PDE) together with (BC1,BC2,BC4) as before and (BC3) replaced by (BC3a) \( u(\pi,y) = f(y) \) for \( 0 \leq y \leq \pi \). [[**Hint:** This is very easy if you have completed part a.]]

**Explain in detail** how the superposition principle is used.
3.a. Use a parameterization to directly evaluate the line integral \( \int_C x \mathbf{j} \cdot d\mathbf{R} \) where \( C \) is the part of the parabola \( y = 3x^2 \) from \((1,3)\) to \((2,12)\). Show details.

b. Without using parameterizations, evaluate the line integral \( \int_C y \mathbf{j} \cdot d\mathbf{R} \) where \( C \) is the part of the parabola \( y = 3x^2 \) from \((1,3)\) to \((2,12)\).

c. Let \( C \) be the boundary of the triangle with corners \( A(0,0), A(3,0), \) and \( C(3,6) \) (oriented counterclockwise). Use Green's theorem to rewrite the line integral \( \oint_C x^2 \mathbf{j} \cdot d\mathbf{R} \) as a double integral and evaluate this integral.

4. Let \( S \) be the triangle whose edges connect \((2,0,0),(0,4,0),(0,0,12)\) in this order.
   a. Find an equation for the plane containing the triangle \( S \).
   b. Parameterize the triangle \( S \) and directly evaluate the flux integral \( \iint_S z^2 \mathbf{k} \cdot \mathbf{N} \, dA \).
   
   **Bonus.** Use the divergence theorem to rewrite the flux-integral as the difference of a triple integral and three easier double integrals. Evaluate these.

5.a. Which of the three vector fields appear to be linear?
   b. Which of the vector fields shown appear to be conservative? Explain!
   c. Which of the vector fields shown appear to be divergence free? Explain!
   d. Find a possible formula for each vector field.
   e. If possible, find a potential function for each vector field. If impossible, explain why.

6.a. In which physical setting does the vector field \( \vec{F}(x,y,z) = \frac{-y}{x^2+y^2} \mathbf{i} + \frac{x}{x^2+y^2} \mathbf{j} + 0 \mathbf{k} \) arise?
   b. Calculate the curl of the vector field \( \vec{F} \). Show details of your calculation.
   c. Use Stokes' (or Green's) theorem to evaluate \( \oint_{C_1} \vec{F} \cdot d\vec{R} \), where \( C_1 \) consists of the edges of triangle with corners \( A(3,0,0), B(3,3,0), C(0,3,0) \).
   d. Directly evaluate \( \oint_{C_2} \vec{F} \cdot d\vec{R} \) where \( C_2 \) is the unit circle \( x^2 + y^2 = 1 \) in the plane \( z = 0 \).
   e. Use Stokes' theorem evaluate \( \oint_{C_3} \vec{F} \cdot d\vec{T} \), where \( C_3 \) is the triangle with corners \( P(3,3,2), Q(-3,0,2), \) and \( R(3,-3,2) \). Explain your reasoning in detail (e.g., what is "the surface"?).