# Introduction to Calculus of Vector Fields

These course-notes are a draft and were prepared for a course in fall 2000 at ASU.

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These notes for this class of fall 2000 do **not** contain reviews of most prerequisite material as be easily be found in the class' textbook. Similarly, there are some vector calculus topics, especially routine exercises, where the class will be referred to the textbook.

# Chapter 1

# Vector fields

## 1.1 Introduction

Vector fields are abstractions of very familiar images that convey information using lots of arrows. One example are *force fields* and magnetic field lines that surround are magnet. Another common picture are geographical maps of prevailing winds or currents.

Figure 1.1: A magnetic force field, and a map showing prevailing winds

These examples associate to every point (usually on some grid) an arrow distinguished by its direction and its length. The arrow points in the direction of the force or in the direction into which the wind is blowing at that point. The length of the arrow symbolizes the strength, or magnitude, of the force, or the speed of the wind at that point.

Mathematically a vector field is a special kind of function: The inputs in the examples above are points in the plane. The outputs are vectors – which abstract the pictorial arrows.

The first task is to make mathematically precise this notion of a vector field as a function. Thereby we will gain access to a familiar machinery of algebraic and analytic tools.

Our development shall be a two way street: Mathematical objects provide the tools for modeling physical phenomena. In turn these provide questions and problems which motivate, often even guide, further mathematical constructions.

For a very naive preview consider again the two examples given above. One natural question asks which path a speck of dust will take as it is blown around by the wind. Similarly, one may ask which path a small particle of magnetic material, or an ion, will follow if subject only to the magnetic force field. Mathematically, these lead to two different notions of *integrals of vector fields*. One is commonly studied under the name of *differential equations*, the other is connected to line integrals in vector calculus.
For another question think of the vector field modeling the current in some ocean. A natural question asks how much water will flow every year through the narrows between two islands. Similarly, a fisherwoman may ask how much water will flow through her fishing net during one night. Mathematically, these questions will lead to different notions of integrals of vector fields, different kinds of line integrals, and surface integrals.
Remembering the first calculus course, a natural question is whether there is an analogue of the fundamental theorem of calculus.

**Exercise 1.1.1** *Recall the first and second fundamental theorems of calculus. Look up, and write down each theorem in precise mathematical terms, with all hypotheses. Then write down in plain English what they say. Finally give one example each for how they are used.*

As another motivation, consider the space shuttle as it travels from the ground to its desired orbit around the Earth. A natural question is whether some flight paths use less energy than others as the shuttle is lifted against the gravitational force field of the Earth. Mathematically, this becomes a question about whether every, or some, or no vector field(s) have an antiderivative. (Recall that every (continuous scalar) function (of a single variable) is the derivative of some function. A major thread of vector calculus investigates if this generalizes in some sense to vector fields).
Physically such antiderivative may correspond to a notion of *potential energy*. The associated fundamental theorem for line integrals then has an interpretation in terms of conservation of energy.
This course we will find several more candidates for fundamental theorems of calculus of vector fields. Initially these may appear quite different from one another. The final goal of the course is the triumphant formulation, and use, of the general *Stokes' theorem* that contains all as special cases.

Derivatives of vector fields may arise from questions about rates of change of vector fields. There are quite a few possible directions in which such questions may lead. However, meaningful derivatives of vector fields most naturally arise after first developing useful integrals of vector fields. The *desired* fundamental theorems will provide additional guidance to develop useful notions of derivatives! This is much more natural than the experience of first year calculus may suggest: Integration is about multiplying and adding, differentiation about differences and quotients. Arguably the natural order is to first learn addition and multiplication, and then develop subtraction and division by asking e.g.: How much do we have to add to $a$ to get $b$? How many times $a$ is equal to $b$?

**Exercise 1.1.2** *Make a list of at least five different phenomena that may be suitably pictured by an arrow at every point. Try to make as diverse a collection as possible. Think broadly, go beyond physics, and consider e.g. migration in biology and social studies,* gradients *of temperatures or of concentrations of chemicals, and others …*

**Exercise 1.1.3** *For each of your examples in exercise 1.1.2 formulate one or two practical questions that might have answers in terms of integrals or derivatives.*

## 1.2 Vector fields as functions

This section makes the transition from intuitive pictures to precise mathematical language. Recall the starting point are physical phenomena that intuitively may be visualized by collections of arrows, one at each point. We want a formal notion of *vector fields* that is suitable to *model* such phenomena.

The key observation is that at every point there should be exactly one arrow. This means that the points are the inputs of a function – henceforth called vector field – and the arrows are the outputs.

The points may be restricted to lie on some grid. But in many applications – think of the wind and the magnetic field – the speed and force make sense at all points. The arrows are drawn at a few selected points only to prevent hopeless clutter. Thus natural *domains* for the vector fields we have in mind are (subsets) of the plane and of three dimensional space. Identifying points with their coordinates, i.e. pairs or triples of numbers allows for simple precise descriptions of the inputs.

For the outputs we abstract the *pictorial arrows* to *mathematical vectors*. There are many different ways, and levels, to think about vectors. On an intuitive level vectors are almost the same as arrows, namely *quantities* that have a direction and a magnitude. This description is particularly useful to provide guidance in physical applications in two- and three dimensional space. Upon choosing a set of *basis vectors*, each vector may then be identified with its coordinates, i.e. a pair or triple of numbers. Natural first choices for such *bases* are the vectors of unit length from the origin and pointing in the direction of the (positive coordinate) axes. In the physical sciences these are commonly denoted by $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$. Alternative notations include column vectors

$$\vec{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{\jmath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and in 3-space} \quad \vec{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.1}$$

Other places use $e_1$, $e_2$ and $e_3$ to denote the same vectors. For example the *arrows* based at the point $P(2,1,3)$ with head at $Q(-1,5,3)$ may then be denoted by either

$$\vec{v} = -3\vec{\imath} + 4\vec{\jmath} \quad \text{or} \quad \vec{v} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} \tag{1.2}$$

Note the ambiguity in the first expression which does not make it clear whether $\vec{v}$ is a vector in the plane or in 3-space! Throughout this introduction to vector calculus we will use either notation as is convenient. One good motivation for this is that on one side the $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ is very firmly established in the sciences. On the other side, most any computer program simply works with pairs or triples of numbers.

**Exercise 1.2.1** *Write each of the following vectors in the form* $a\,\vec{\imath} + b\,\vec{\jmath} + c\,\vec{k}$*, and as a column vector.*

a. *The vectors along the sides of the triangle with vertices at* $(-1, 0)$, $(1, 0)$ *and at* $(0, \sqrt{3})$.

b. *A vector of unit length that is tangent to the graph of* $y = x^2$ *at the point* $(1, 1)$.

c. *A vector in the plane that has unit length and is perpendicular to tangent line to to the graph of* $y = x^2$ *at the point* $(1, 1)$.

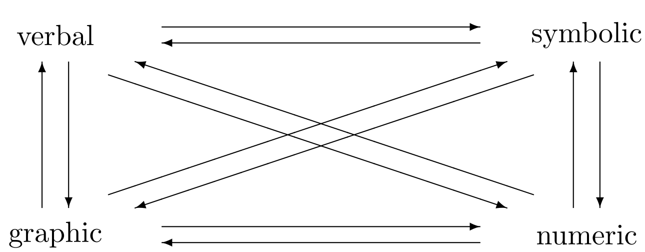
d. *A vector of unit length that is perpendicular to the triangle with vertices at* $(4, 1, 1)$, $(1, 5, 1)$ *and* $(1, 1, 6)$.

*On the side we note that once one makes the step from arrows to pairs or triples of numbers, it is natural to consider* $n$*-tuples of numbers, i.e. generalize vectors to* $n$ *dimensions. This way a collection of* $n$ *measurements or data points may be considered a vector, and a small step further, functions are considered vectors in infinite dimensional spaces. The formal development of this mathematical notion of vectors is studied in linear algebra. In this introduction to vector calculus hardly any of this is needed – only occasionally shall we appeal to a little open-mindedness, e.g. to explore whether our constructions make sense in general settings, or are restricted to very special e.g. two dimensional phenomena. For the sake of completeness, we note that mathematically , vectors are – loosely speaking – objects for which sums and scalar multiples (that obey the* usual rules*) make sense. More precisely, vectors are the elements of a vector space, which is a* set *together with two* operations*, called addition and scalar multiplication, which satisfy a list of basic* axioms.

Thus we have a preliminary definition:

**Definition 1.2.1** *A vector field is a function that assigns to every point in (some subset of the) plane or* 3*-space a unique vector in the plane* 3*-space, respectively.*

Just like functions that assign a number $y$ to every number $x$ (possibly in some subset of numbers), vector fields may be communicated in different ways: The first examples where given by pictures. For practical computations tables of numbers are often the most suitable way. For symbolic calculations algebraic formulas are generally desirable. Finally, very powerful are verbal descriptions such as: *"Let* $\vec{F}$ *be the vector field in* 3 *space that is defined everywhere except at* $(0, 0, 0)$*, that has unit length and points towards* $(0, 0, 0)$ *at every point."*




One of the most basic, essential skills is to go from any one of these descriptions to any other. Some of these directions have the usual ambiguities, e.g. any finite collection of pictured arrows or numerical data can be fitted exactly with infinitely many different formulas.
Clearly we do not have the space to discuss in detail all twelve directions. But rather than despair, note that often it is convenient to go an indirect route, e.g. from a verbal description, to graphical sketch, then read off a number of specific values, and use these to guess a possible formula.
In the following we consider selected translations between these descriptions.

**Example 1.2.1** *An infinitely long straight wire carrying a constant electric current generates a magnetic vector field. The magnitude of the field is inversely proportional to the distance from the wire. The field is at every point perpendicular to both the wire and the shortest line from the point to the wire. Its direction is further governed by the* right hand rule*: Roll up the fingers to make a fist, but leave the thumb extended. If the thumb points in the direction of the current, then the fingers indicate the direction of the magnetic field.*

*Translate this description into an analytic formula.*

**Example 1.2.2** *Find a formula for the magnetic field described in exercise 1.2.1.*

We are looking for three scalar functions $M_1$, $M_2$ and $M_3$ of three variables each such that $\vec{M}(x,y,z) = M_1(x,y,z)\vec{\imath} + M_2(x,y,z)\vec{\jmath} + M_3(x,y,z)\vec{k}$ is everywhere perpendicular to the wire, and such that the magnitude $|\vec{M}(x,y,z)|$ is inversely proportional to the distance of the point $(x,y,z)$ from the wire.
To simplify matters, choose coordinates such that the $z$-axis is aligned with the wire. To be perpendicular to the wire, the vector field must satisfy

$$0 \stackrel{!}{=} \vec{k} \cdot \vec{M}(x,y,z) = M_3(x,y,z) \quad \text{for all} \quad x,y,z. \tag{1.3}$$

This leaves only two functions left to be determined. The shortest line segment from the wire to a point $P = (x,y,z)$ is described by the vector $\vec{r} = x\vec{\imath} + y\vec{\jmath}$. The condition that $\vec{M}(x,y,z)$ be perpendicular to this vector and to the wire means that $\vec{M}(x,y,z)$ must be a multiple of the cross-product $\vec{k} \times \vec{r}$, i.e.

$$\vec{M}(x,y,z) = m(x,y,z)\vec{k} \times \vec{r} = m(x,y,z)\ (y\vec{\imath} - x\vec{\jmath}) \tag{1.4}$$

for some scalar function $m$ still to be determined. The right hand rule forces that $m$ is always positive.
The magnitude of the field at any point $(x,y,z)$ is inversely proportional to the distance $|\vec{r}| = \sqrt{x^2+y^2}$, i.e. for some constant $C > 0$

$$|\vec{M}(x,y,z)| = m(x,y,z)\,|y\vec{\imath} - x\vec{\jmath}| = m(x,y,z)\,\sqrt{x^2+y^2} \stackrel{!}{=} \frac{C}{\sqrt{x^2+y^2}} \tag{1.5}$$

Thus $m(x,y,z) = C\,(x^2+y^2)^{-1}$ and finally

$$\vec{M}(x,y,z) = \frac{-y\,\vec{\imath}}{x^2+y^2} + \frac{x\,\vec{\jmath}}{x^2+y^2} \quad \text{for all} \quad (x,y,z) \neq (0,0,*) \tag{1.6}$$

where we have chosen the scaling constant $C = 1$. This formula is well worth memorizing as this vector field is arguably one of the most important fields in physical applications. Moreover, as we shall see soon, it is also one of the most interesting fields mathematically.

Given a formula for a vector field in the plane it is very easy to obtain a table of values, and use this to obtain a graphical image. Today, most computer programs combine these steps, apparently obviating the need to first create a table. However, without some practice by hand, the machine's plots are easily conceived as magic. Moreover, every once in a while the machines don't work as expected, have bugs, or show some unexpected results. In such cases it is an essential skill to be able to verify the correctness of the machine plot, or to track down the origin of the trouble.

**Exercise 1.2.2** *Use your favorite computer program to plot the vector field* $\vec{M}(x, y) = \sin(12xy)\vec{\imath} + \cos(11x^2 - 9y)\vec{\jmath}$*. Explain in words why, say on the rectangle* $[-10, 10] \times [-10, 10]$*, the image appears to be very complicated.*

**Example 1.2.3** *Make a table of function values for the magnetic field discussed above, and sketch a suitable cross-section of the vector field.*

Since the vector field $\vec{M}$ depends only on the first two coordinates and is horizontal, it may also be considered as a vector field in the plane! As a function of points in the plane, i.e. of pairs of numbers, we need a two-dimensional, rectangular table. To facilitate going back and forth from pictures to tables, it is convenient to arrange the $x$-values from left to right, and the $y$-values from bottom to top. Evaluating the component functions $M_1 = \vec{\imath} \cdot \vec{M}$ and $M_2 = \vec{\jmath} \cdot \vec{M}$ at the grid points, yields the following table of approximate values

| | $-2$ | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $2$ |
|---|---|---|---|---|---|---|---|
| $2$ | $-.25\vec{\imath} - .25\vec{\jmath}$ | $-.40\vec{\imath} - .20\vec{\jmath}$ | $-.47\vec{\imath} - .12\vec{\jmath}$ | $-.50\vec{\imath} + 0.0\vec{\jmath}$ | $-.47\vec{\imath} + .12\vec{\jmath}$ | $-.40\vec{\imath} + .20\vec{\jmath}$ | $-.25\vec{\imath} + .25\vec{\jmath}$ |
| $1$ | $-.20\vec{\imath} - .40\vec{\jmath}$ | $-.50\vec{\imath} - .50\vec{\jmath}$ | $-.80\vec{\imath} - .40\vec{\jmath}$ | $-1.0\vec{\imath} + 0.0\vec{\jmath}$ | $-.80\vec{\imath} + .40\vec{\jmath}$ | $-.50\vec{\imath} + .50\vec{\jmath}$ | $-.20\vec{\imath} + .40\vec{\jmath}$ |
| $\frac{1}{2}$ | $-.12\vec{\imath} - .47\vec{\jmath}$ | $-.40\vec{\imath} - .80\vec{\jmath}$ | $-1.0\vec{\imath} - 1.0\vec{\jmath}$ | $-2.0\vec{\imath} + 0.0\vec{\jmath}$ | $-1.0\vec{\imath} + 1.0\vec{\jmath}$ | $-.40\vec{\imath} + .80\vec{\jmath}$ | $-.12\vec{\imath} + .47\vec{\jmath}$ |
| $0$ | $+0.0\vec{\imath} - .50\vec{\jmath}$ | $+0.0\vec{\imath} - 1.0\vec{\jmath}$ | $+0.0\vec{\imath} - 2.0\vec{\jmath}$ | undefined | $+0.0\vec{\imath} + 2.0\vec{\jmath}$ | $+0.0\vec{\imath} + 1.0\vec{\jmath}$ | $+0.0\vec{\imath} + .50\vec{\jmath}$ |
| $-\frac{1}{2}$ | $+.12\vec{\imath} - .47\vec{\jmath}$ | $+.40\vec{\imath} - .80\vec{\jmath}$ | $+1.0\vec{\imath} - 1.0\vec{\jmath}$ | $+2.0\vec{\imath} + 0.0\vec{\jmath}$ | $+1.0\vec{\imath} + 1.0\vec{\jmath}$ | $+.40\vec{\imath} + .80\vec{\jmath}$ | $+.12\vec{\imath} + .47\vec{\jmath}$ |
| $-1$ | $+.20\vec{\imath} - .40\vec{\jmath}$ | $+.50\vec{\imath} - .50\vec{\jmath}$ | $+.80\vec{\imath} - .40\vec{\jmath}$ | $+1.0\vec{\imath} + 0.0\vec{\jmath}$ | $+.80\vec{\imath} + .40\vec{\jmath}$ | $+.50\vec{\imath} + .50\vec{\jmath}$ | $+.20\vec{\imath} + .40\vec{\jmath}$ |
| $-2$ | $+.25\vec{\imath} - .25\vec{\jmath}$ | $+.40\vec{\imath} - .20\vec{\jmath}$ | $+.47\vec{\imath} - .12\vec{\jmath}$ | $+.50\vec{\imath} + 0.0\vec{\jmath}$ | $+.47\vec{\imath} + .12\vec{\jmath}$ | $+.40\vec{\imath} + .20\vec{\jmath}$ | $+.25\vec{\imath} + .25\vec{\jmath}$ |

Since the vector field looks the same in any horizontal plane, it suffices to sketch it in the $xy$ plane. It is straight forward to draw at each grid point $(x_i, y_j)$ of the above table the corresponding vector $\vec{M}(x_i, y_j, *)$. The picture shown is a computer generated plot that simply fills more intermediate plots.

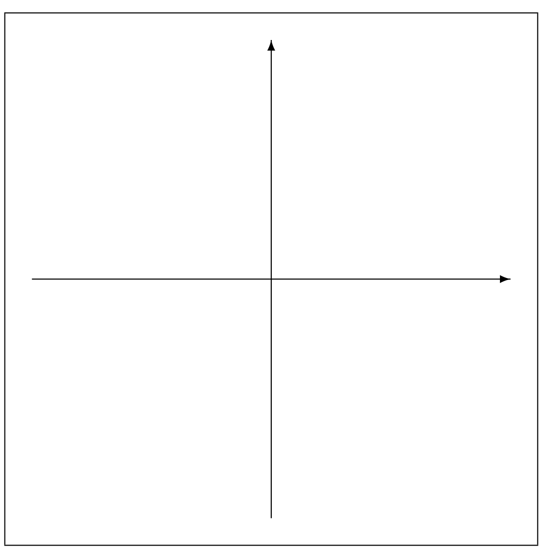


Figure 1.2: The magnetic field about a straight wire carrying a constant current

Care needs to be taken as many computer plotting programs will be unable to plot the field due to the division by zero at $(x, y) = (0, 0)$. Mathematically the vector field is undefined at $(0, 0)$ – and it may take some effort to enter this into the plotting program. Physically, the points $(0, 0, z)$ correspond to the inside the wire where the magnetic field is ????, is it zero inside?.

**Exercise 1.2.3** *For each the following vector fields create a table of values, and use it to sketch by hand an image of the field. Then check your result against a plot using a computer program.*

- $\vec{S}(x, y) = \vec{\imath} + x^2\vec{\jmath}$
- $\vec{H}(x, y) = -y\vec{\imath} + x\vec{\jmath}$
- $\vec{G}(x, y) = (x\vec{\imath} + y\vec{\jmath})/(x^2 + y^2)$

In many cases one can directly sketch the vector field from a verbal description without having to go through formulae and tables – but that detour always is an option.

**Exercise 1.2.4** *Sketch a vector field in the plane that has the same magnitude everywhere and points everywhere in the direction of the positive $x$-axis.*

**Exercise 1.2.5** *Sketch a vector field in the plane that points everywhere in the direction of the positive $y$-axis and whose magnitude is proportional to the distance from the $x$-axis.*

**Exercise 1.2.6** *Sketch a vector field in the plane that points everywhere in the direction of the positive $y$-axis and whose magnitude is proportional to the distance from the $y$-axis.*

**Exercise 1.2.7** *Sketch a vector field in the plane that at every point $P$ is perpendicular to the vector $\vec{OP}$ from the origin to $P$, and whose magnitude is proportional to the distance from the origin.*

**Exercise 1.2.8** *Sketch a vector field in the plane that at every point $P$ is perpendicular to the vector $\vec{OP}$ from the origin to $P$, and which has unit magnitude.*

**Exercise 1.2.9** *Sketch a vector field in the plane that at every point $P$ points towards the origin, and whose magnitude is proportional to the distance from the origin.*

**Exercise 1.2.10** *Sketch a vector field in the plane that at every point $P$ points towards the origin, and which has unit magnitude.*

**Exercise 1.2.11** *Sketch a vector field in the plane that at every point $P$ points towards the origin, and whose magnitude is inversely proportional to the distance from the origin.*

There are many other ways to go from one to another representation of a vector field. Just like reading a graph of a scalar function, it is a basic skill to read off values of a vector field and make a table when given a picture.

**Exercise 1.2.12** *Consider the vector field pictured in figure 1.1. Choose a coordinate system and make a table of values of the pictured vector field.*

Another valuable skill is to be able to precisely describe in words the key features of a pictured vector field. This takes practice. As examples for concise verbal description consider the set of exercises above which go in the other direction.

**Exercise 1.2.13** *Add several pictures*

Another direction that takes practice is visualize a vector field from just a brief look at a table of numerical values. We will practice this in the upcoming section on linear vector fields.
Some of the hardest translations are finding possible formulas that match a picture or table of values. For us this is important primarily in a few special cases – just like in the first calculus one should immediately recognize the shapes of a few basic function graphs! Here the most important ones are constant and linear fields, and the special fields of primary physical interest, the magnetic field discussed above, and the gravitational (or electro static) field about a point mass (charge). Linear fields will be discussed in detail in a subsequent section, allowing for practice of matching formulas to pictures and tables.
We conclude this section with a few exercises to go from formulas to pictures to words.

**Exercise 1.2.14** *Plot each of the following vector fields using a computer program. Then describe this vector fields in words as possible. Pay special attention to the changes of magnitude, the direction (e.g. horizontal radially outwards etc.), and whether the horizontal and vertical components depend on $x$, $y$ or whatever else. Be creative, but precise. Think of giving instructions via telephone to a friend who needs to draw an accurate copy of a pictured vector field that is in your hands. Pretend that you do not have a formula available.*

- $\vec{F}(x,y) = \vec{\imath} - 2\vec{\jmath}$
- $\vec{G}(x,y) = \sqrt{x^2+y^2}(\vec{\imath} + \vec{\jmath})$
- $\vec{H}(x,y) = -y\vec{\imath} + x\vec{\jmath}$
- $\vec{K}(x,y) = (y\vec{\imath} + x\vec{\jmath})$
- $\vec{U}(x,y) = (-y\vec{\imath} + x\vec{\jmath})/\sqrt{x^2+y^2}$

## 1.3 Applications and different kinds of vector fields

This section takes a closer look at different kinds of vector fields that arise in different contexts. All of these are customarily pictured in the same way. However, guided by practical questions arising from applications we shall see that one should distinguish several different categories of vector fields. This course shall just initiate some awareness about the differences. The main goal is to develop some intuition about which kinds of integrals and which derivatives are meaningful in which contexts. The precise mathematical distinctions are subject of advanced classes in e.g. differential geometry. For a lovely pictorial examination of almost a dozen different kinds of vector fields see G. Weinreich's *"Geometrical Vectors"* (University of Chicago Press?).

## 1.4 Velocity fields and differential equations

One of the most intuitive uses of vector fields is to model velocity fields. These may involve rigid bodies, large numbers of particles (including elementary particles), fluids and gases.

For example consider a rigid body rotating with with constant angular velocity $\omega$ about the $z$-axis. (This means that the time it takes to complete one full revolution is $T = \frac{2\pi}{\omega}$, also called the *period*.) At any time the velocity of a point $(x, y, z)$ on the rotating body is $\vec{v} = \omega(y\vec{\imath} - x\vec{\jmath})$.

Take a quick look at the dimensions: The coordinates $x$ and $y$ are lengths, the angular velocity has the dimension of $\frac{1}{\text{time}}$, and thus the vector field has the expected dimension of a velocity, distance per time, e.g. meters per second. Now suppose one changes the units, say from meters to centimeters. Then the numerical values of the coordinates $x$ and $y$ of the same point increase by a factor of 100, and so does the numerical value of the velocity of this point.
Associated with this vector field is a system of ordinary, autonomous differential equations:

$$\begin{aligned} \frac{dx}{dt} &= \omega y & (= \vec{\imath} \cdot \vec{v}) \\ \frac{dy}{dt} &= -\omega x & (= \vec{\jmath} \cdot \vec{v}) \end{aligned} \tag{1.7}$$

The general solution $x(t) = x_0 \cos \omega t + y_0 \sin \omega t$, $y(t) = -x_0 \sin \omega t + y_0 \cos \omega t$ describes the path traced out by the point on the rigid body that was located at $(x_0, y_0)$ at time $t = 0$.

**Exercise 1.4.1** *Plot the vector field $\vec{v}$ and solution curves $(x(t), y(t))$ for different initial points $(x_0, y_0)$. Algebraically verify that the curves are indeed circles by simplifying $(x(t))^2 + (y(t))^2$. Describe the relation of the vector field $\vec{v}$ and the curves $(x(t), y(t))$.*

Recall that the solution curves of differential equations are also called *integral curves*. A little reflection should lead one to agree that this is a well chosen name: Going from the velocities to the positions basically means integrating with respect to time. Thus integrating a system of differential equations is a first kind of integral for vector fields!

Note that there is nothing special about this vector field. Indeed, every vector field determines a system of autonomous differential equations. Conversely, every system of autonomous differential equations may be identified with a vector field. Thus in some sense one may claim that differential equations and vector fields are the same. Indeed many mathematicians only work with vector fields, which as functions have *better* properties than *equations*.
However, viewing a vector field just as a system of differential equations is not always the most useful approach. The upcoming examples of force fields shall underline this reservation.

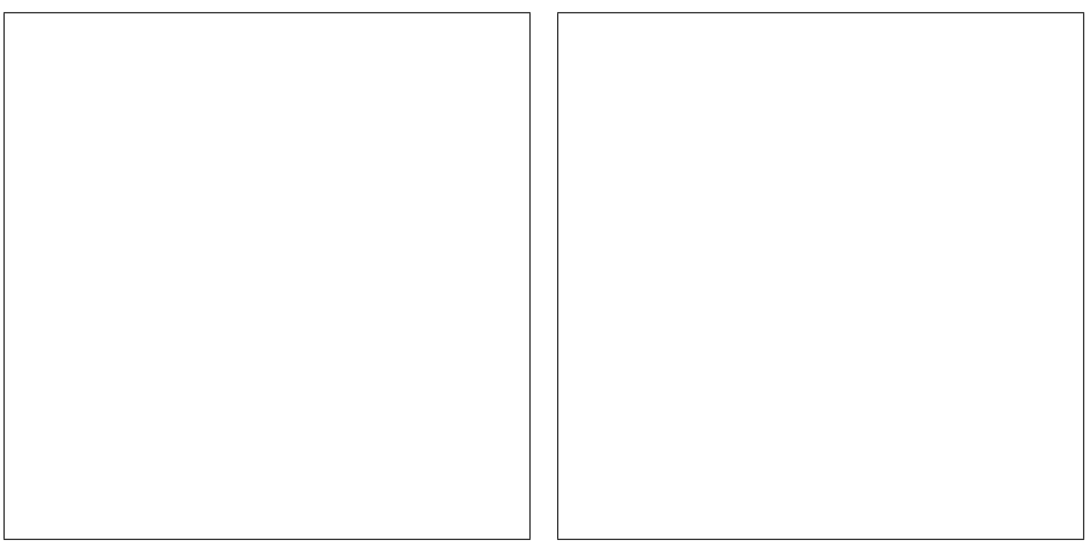


Figure 1.3: Fluid flow around a cylinder and in a rocket exhaust

Returning to the velocity field, now consider a fluid in motion. An interesting case is water flowing past an obstacle which may be as large as an island, or as small as a screw protruding

into a pipe. Another rewarding example is the water (or blood) flowing through a pipe or blood vessel that has a constriction (e.g. a calcium deposit) or that has an enlarged section (e.g. an aneurysm)
Questions about modeling and different notions of solutions of fluid flows have been some of the hottest research areas in recent decades, and no final solution is in sight. Thus it is natural for us to consider very special cases, making simplifying assumptions. Our first assumption is that the flow does not show any turbulence, characterized by *eddies* of all sizes with a *chaotic* appearance. We shall also ignore *boundary layer* effects which basically model that the fluid *sticks* to the surfaces in some sense.
Simple formulas that model such fluid flows may be obtained fairly easily using functions of a complex variable.

$$\vec{V}(x,y) = \boxed{\text{Flow around cylinder}} \qquad \vec{W}(x,y) = \boxed{\text{Flow thru constriction}} \tag{1.8}$$

These models for two dimensional flows have the obvious advantage that they are more easily to picture. Almost all physical flows are three dimensional. But the two dimensional models may still be useful to describe flows in thin films, or shallow water as e.g. in the Everglades or in a flooded plane. For us most important is that such two dimensional models already capture many phenomena that are of primary interest to us.

Just as before for the rigid body, one can solve the associated system of differential equations and thus obtain the curves along which idealized small fluid elements flow. But we may also consider different kinds of *integrals*. One of the principal characteristics of fluids like water is that they are virtually incompressible. (This makes fluids useful for example in hydraulic braking systems in a car!) An fundamental conservation principle thus asserts that the volume entering the pipe on one end must be equal to the volume exiting from the pipe on the other end. Let us take a look at the dimensions – again in the case of a three dimensional pipe: Suppose the vector field determines the velocity in units of centimeters per second. The cross-section of the pipe might be measured in square centimeters. The product of these dimensions yields cubic centimeters per second, i.e. a volume per time. Suppose the velocity $\vec{v}$ is constant and perpendicular to the *imagined surface* at the end of the pipe. In this case it appears reasonable that the volume per time passing through the end is obtained by simply multiplying the magnitude $|\vec{v}|$ of the velocity with the cross-section area. In general, we need to account for velocities that are not perpendicular to the *imagined surface*. Here vector algebra and dot products shall come in handy to account for the angles. Integral calculus shall serve to account for curved surfaces and variable velocity fields $\vec{v}$. There is a lot of work ahead to fill in the technical details, but this exploration is the beginning of a different kind of integral of vector fields: The flux of a vector field *across* a surface, and in the two-dimensional analog, the flux across a curve.

We conclude this discussion of velocity fields with a few remarks about gases and particles. The suggestive example of a gas exiting from a rocket engine dramatizes the main difference between fluids and gases: Gases can be compressed substantially, and they also expand dramatically when heated. Thus, back in the picture of the flow through a pipe, the volumes entering and exiting may be very different. Indeed, the difference in volume – again available through surface integrals must be attributable to what has happened inside the pipe. This might be heating, some chemical reactions, or due to compression. Nonetheless, mass will be conserved. Thus aside from the velocity field $\vec{v}$ (measured e.g. in meters per second), one might also consider the mass flow $\vec{u}$ in units of e.g. kilograms per square meter per second.

These units may be a little confusing: E.g. when changing the units from meters to centimeters, the magnitude of the vector field apparently would decrease by a factor of 10,000, where we expected an increase by a factor of 100. this mystery is easily resolved by regrouping the units as (kilograms per cubic meter) times (meter per second). When changing the units in the domain of the vector fields from meters to centimeters, one should only change the meters ion the second term. The first should has the dimensions of a mass-density and should be treated as one block. A final scenario is that the vector field models the velocity field of a very large number of individual particles. Think of huge numbers of electrons or photons. Again the associated differential equations yields the curves along the particles move. Nothing new here. The surface integral shall provide the number of particles that cross an *imagined surface* per unit of time. Thus the dimensions of the vector field should be number of particles per area per unit of time. Again, more intuitive might be (number of particles per unit volume) times (distance per time). In the case of charged particles, the (total charge per unit volume) takes the place of the (number of particles per unit volume).

| turbulence, vorticity, and line integrals!! |
|---|

## 1.5 Gradient fields

In multi-variable calculus vector fields in the plane and in 3-space appear as derivatives of scalar functions of two and three variables. The *gradient* of a function $z = f(x, y)$ of two variables is the vector field

$$\vec{F}(x, y) = \frac{\partial f}{\partial x}(x, y)\,\vec{\imath} + \frac{\partial f}{\partial y}(x, y)\,\vec{\jmath}. \tag{1.9}$$

Intuitively consider the graph of the function as modeling a hilly landscape. Then the gradient points at any point $(x, y)$ in the direction of the steepest increase of the graph, and the magnitude equals the slope in this direction. Another important property of the gradients are perpendicular to the contours, or level curves, of the function $f$. In order to see this perpendicularity on a graph it is essential to use equal scales along the coordinate axes!

In a typical application such a function of two variables might model the temperature at different points of a planar region. Then the gradient points at any point in the direction in which the rate of change of temperature is the greatest, and the magnitude is equal to this rate of change of temperature. Regarding the units, suppose that $x$ and $y$ determine the distances in meters from fixed coordinate axes, and that the temperature $z$ or $T$ is measured in degrees Celsius. Then each component of the gradient, and thus also its magnitude, give the rate of change of temperature in degrees per meter. If one now changes the units from meters to centimeters, the $x$ and $y$ coordinates of the same point each would be 100 times larger then the original coordinates. However, the rate of change of the temperature, and thus the magnitude of the vector field would *decrease* by a factor of 100. This is the *opposite* of the situation with the velocity fields discussed in the preceding subsection.

This observation strongly suggests that there are some fundamental differences between different kinds of vector fields. An in-depth analysis, with very precise terminology is the subject of advanced courses in differential geometry, and far beyond the scope of this course. Nonetheless it will often be helpful in even the studies at our level to categorize a vector field as *like a velocity field* or *like a gradient field*. The criterion is to just take a brief look how the vector field transforms under changes of scale, which are some of the most simple coordinate changes.

The technical names for these kinds of vector fields are *contravariant* if the changes are in the *same* direction – like in a velocity field, and *covariant* if the changes are in the opposite direction (like in gradient fields). These names are very unfortunate choices, but – the reversed names would make more sense. But they have such a long tradition that today nobody has the guts to champion the change . . .

Let us take a brief look at questions about gradient fields that may lead to notions of *integrals* (and possibly, derivatives) of vector fields. Clearly one always can write down a system of differential equations

$$\begin{aligned} \dot{x} &= -\tfrac{\partial f}{\partial x}(x,y), \quad \text{and} \\ \dot{y} &= -\tfrac{\partial f}{\partial y}(x,y). \end{aligned} \tag{1.10}$$

(be patient regarding the minus sign). The question is whether these differential equations, or their solutions have any practical values. Consider first the case in which the function $z = f(x, y)$ is describes a real hilly landscape. In this case a very suggestive interpretation is that the solution curves of the differential equations (with the minus sign) describe the path water (a creek) running down the mountain will take. However, this is only correct in a rough approximation as water will flow up-hill over obstacles, provided it arrives with sufficiently speed! Thus on a small scale, of at most a few meters and very fast rushing water the solution curve of the differential equations does *not* give the correct path of the water. However, on a sufficiently large scale, with elevations changes of hundreds, or thousands of meters, and typical water velocities as in rivers and streams, the solution curve provides a very good model for the path of the stream.
A simple explanation of this surprise is that the gradient field is not a velocity field, but rather almost an acceleration field: The steepness of the mountain is intimately related to the effect of the gravitational force, which in turn, via Newton's law translates into accelarations. The next subsection will further follow up on this.
Mathematically the system differential equation (1.10) is very useful for numerically calculating (local) minima of a function $z = f(x, y)$. The idea is very simple: Calculate the gradient of the function $f$, and numerically solve the differential equations (1.10) with many randomly chosen initial conditions. If the function $f$ is bounded from below, every solution curve will eventually reach an equilibrium which must be a critical point of the function. The simplicity of this approach lends it much robustness – and with many technical improvements this idea leads to very efficient numerical optimization algorithms. but this belongs into numerical analysis, not into vector calculus.

Thus for gradient fields the *integration* of the system of differential equations (1.10) does not appear to solve very compelling problems about vector fields.
Similarly, one may explore whether there is any good use for an analogue of the *flux integrals* discussed for velocity fields. This would lead to descriptions like: *"How much steepness flows across a given curve?"*, *"How much temperature flows across a curve or through a surface?"*. Neither one sounds very exciting – indeed, unless a gradient field also can be suitably interpreted as velocity field, or some similar object, we claim that such integrals of gradient fields are in general rather meaningless!

However, as a derivative of a function, it is natural to ask whether, and how, one may recover the function from the gradient field! If possible, this should be considered a way of *integrating* vector fields. The idea is actually very simple: If the gradient field tells us at every point how the graph of the original function is tilted, then one should be able to reconstruct changes in

altitude from the distance traveled together with the information of the slope. This is very compelling in the special case where the original graph is a plane, and the path is a straight line. Next it is intuitive that one should be able to put seevarl such planar-linear pieces together. Finally calculus comes in – if the function $f$ is differentiable – its graph is on a sufficiently small scale arbitrarily close to a plane. Similarly, a differentiable path is on a sufficiently small scale arbitrarily close to a straight line. With this in mind it will be rather straight forward to construct *line integrals* – which will return the original function. Given a reference height at one point, one can calculate the function value at any other point simply by adding the elevation change along any path from the reference point to the desired point.
Briefly revisit the case of the gradient field of a function modeling temperature in a planar region. In this case, integrating the rate of change of temperature along any path will yield the total change of temperature between the starting and end point of the curve.
This general description should appear very familiar from first year calculus. The powerful property of line integrals of gradient fields returning the total change in height, temperature, ... deservedly will be cast as the *fundamental theorem of line integrals.*

## 1.6 Not every vector field is a gradient field

Recall from first year calculus that every (continuous) scalar function $f$ of a scalar variable is the derivative of some function $F$. In other words, every function has an antiderivative. As taught in calculus, for many functions, such as $f(x) = e^{-x^2}$, $g(x) = \frac{\sin x}{x}$, $h(x) = \cos x^2$ the antiderivative cannot be expressed in terms of the familiar elementary functions. Nonetheless, each of these functions has an antiderivative. If

$$F(x) = \int_0^x e^{-t^2} dt, \quad G(x) = \int_0^x \frac{\sin(t)}{t} dt, \quad \text{and} \quad H(x) = \int_0^x \cos t^2 \, dt \tag{1.11}$$

then $F' = f$, $G' = g$ and $H'h$. The functions $F$, $G$, and $H$ are well defined, in a very constructive way. For any value of $x$ e.g. very basic numerical algorithms allow one to calculate the function value to arbitrary accuracy.

Figure 1.4: Could either of these vector fields be a gradient field?

Consider the vector field $\vec{H}(x,y) = (y\vec{\imath} + x\vec{\jmath})$ in figure 1.6. If it was a gradient vector field, then the vectors would indicate the direction of steepest increase. Consider traveling along the unit circle, starting and ending at the point $(1,0)$. At every point along this curve the vector field is tangent to the path (in the forward direction!). Thus the curve is always climbing as steeply as possibly. But at the end the path returned to its starting point – and the total change in *elevation* must be zero – if the vector field was a gradient field. However, since the line integral is definitively strictly positive this leads to a contradiction. Consequently, $\vec{H}$ cannot possibly be the gradient of any function!

Figure 1.5: Artistic renderings by Escher, suggestive of *not gradient*

The famous pictures by Escher in figure 1.6 is a wonderful artistic rendering of this argument.

**Exercise 1.6.1** *Consider the* magnetic field $\vec{M}(x,y) = (y\vec{\imath} + x\vec{\jmath})/(x^2 + y^2)$ *pictured in figure 1.6. Again pretending that this might be a gradient field, analyze the* hypothesized *total change in elevation along closed curves that consist of two circular arcs with radii* $r_1$ *and* $r_2$*, and two radial line segments along lines that form angles* $\Theta_1$ *and* $\Theta_2$ *with the positive* $x$*-axis.*
*Recall that the magnitude of* $\vec{M}$ *is inversely proportional to the the distance from the origin. Use this, the tangency / perpendicularity of the vectors to the curve, and the length of the curve segments to explain why along any such closed curve the total change in elevation is zero. (Thus, the analysis along these curves would be consistent with the vector field being a gradient field.)*
*On the other hand verify that the argument made for the vector field* $\vec{H}$ *along the unit circle still works. Conclude, that even though along many closed curves there does not appear any conflict with the supposition that* $\vec{M}$ *is a gradient field, the argument along the circle shows that it is not a gradient field.*

Thus a simple picture compellingly shows that not every vector field is the gradient field of some function. Indeed *almost every* vector field is *not* a gradient field. Nonetheless, the few vector fields that are gradient fields have very important physical applications. Thus a detailed study of when a vector field is a gradient field is well worth the time and effort.
Later we will establish a very simple algebraic criterion that decides whether a vector field is a gradient field, or not. Nonetheless, we claim that the picture above, and some further

explorations of this situation, provide a much deeper understanding of what is going on. Indeed, this discussion here is just the starting point for a lot of wonderful mathematics, most notably algebraic topology. Also many of the intriguing features of complex analysis can be traced back to the above argument about this simple picture of $\vec{M}(x, y)$.

Traditionally vector calculus has emphasized the study of vector fields that are gradient fields, largely because of their major importance in physics (compare the next subsection). However, the *defect* of not being a gradient field has just as important applications. Arguably one of the most important applications is in thermodynamics and models the process by which machines (generators) convert (differences in) heat into mechanical energy. Thus every steam engine, every gasoline burning car engine, and every electric power plant based on burning some fuel, whether nuclear or fossil, works because of some line integral along a closed curve being nonzero. Indeed, the vertical gap in the spiraling staircase picture of the preceding discussion is precisely the amount of energy converted!

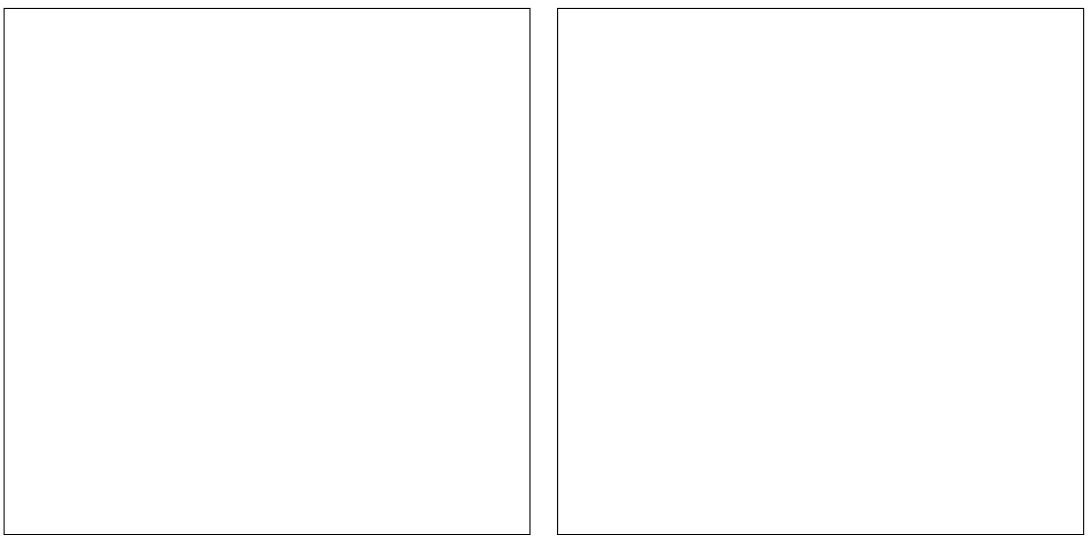


Figure 1.6: Converting (differences in) heat into mechanical energy?

The vector field underlying any such machine is extremely simple $\vec{F}(x, y) = y\vec{\imath}$, as graphed above together with two slightly different closed curves. In theromodynamics this vector field is hidden behind the formula $dU = p\,dV$. Here $p$ stands for pressure, $V$ for volume, and the formula, very loosely stated says that the (infinitesimal) change of internal energy is equal to the product of pressure and (infinitesimal) change in volume. This fundamental relationship makes precise the intuitive notion that one needs to *do work* when compressing a gas – and this work depends on the pressure and on how much the volume is changed. The closed curves shown above model different idealized machines (the Carnot process and Stirling machines) – which really are not that different from car-engines. In any four-cycle car engine the key process involves a piston moving up and down in a cylinder, in which gas is compressed and expanded at different temperatures and pressures. More about this later.

To emphasize that this is not an isolated use, we cursory mention e few other applications. The vertical gap in the spiraling staircase picture of the preceding discussion, or the *defect* of not being gradient is also the key mechanism in control theory. Simple examples include parallel parking a car, re-orienting a satellite by small oscillations of its solar panels, an astronaut rotating herself in space, or a diver performing unimaginable acrobatics. These line integrals even explain how a cat can land on its feet, even when dropped with feet up (with zero total angular momentum). These latter examples are a topic of an accompanying month- or semester long project.

Summarizing, unlike in functions in first year calculus, not every vector field is a derivative (gradient field). The mechanism behind this *defect* is apparent already in very simple pictures. While gradient fields play an very prominent role in classical physics, the defect of being gradient has at least as important applications. Thus developing a deep and comprehensive understanding of being gradient or not is a major goal in vector calculus.

## 1.7 Force fields

Vector fields provide a unifying tool to model such diverse phenomena as magnetized materials that attract iron objects, electro-static attraction repulsion, the reason why objects fall down, and the *glue* that keeps atomic nuclei from flying apart. There have been countless different descriptions and explanations of why objects fall down, many of these predating our recorded history. The effects of magnetic minerals, *lode stones*, that can attract or repel each other have been the subject of much curiosity over a few thousand years [check this ???]. Electrostatic effects, e.g. hair standing on end when an electric storm is too close to be comfortable, or resulting from rubbing amber with some cloth have been the subject of various theories for centuries. The inner working of atoms have been investigated only for about 70 years, and are still ongoing.

Vector fields do not explain why these *forces* happen, but they are well-suited to model their effects. They are functions that assign to every point in the plane or in 3-space a direction and a magnitude, just what is needed to model effects of these diverse phenomena. The formal of a *force* dates at least 350 years back to Newton. Recall Newton's third law (or rather, model): *force equals mass times acceleration*. A force field is a mathematical model that assigns to every point in space a vector that models the force a *test object* at that location would *feel*. The gravitational field describes the force (direction and magnitude, at any point) that *acts on* a *unit mass* at that point. An electric field describes the force that acts on a *unit charge*. A magnetic field can be observed by its effect on small lengths of iron, e.g. on a piece of paper that lies on top of a magnetic: The lengths of iron will rotate into a direction parallel (tangent) to the magnetic field lines. Actually even easier to observe is the effect of magnetic fields on moving charges: A magnet close to the surface of a TV-screen or computer monitor causes quite dramatic distortions of the image (which is created by electrons as they collide with the screen.)

Let us consider some questions that are motivated by the daily life and physics inquiries, and which provide the motivation for calculus of vector fields.

First of all, we would like to know more details about the gravitational, electric and magnetic fields that are generated by a mass, a charge, or a current. The easiest case is when the mass or charge is concentrated in a single point. In that case the directions of the fields are quite intuitive, radially outward or inward. However, the reason why the force fields' magnitudes decrease at the rates that they do is not immediately clear. Indeed, it will only be the big integral theorems of vector calculus that really will shed light on why these rates of decay have the values that they have. They will also resolve the key issues of the mass or charge is not concentrated in one point, but distributed over a ball or a sphere, or even an irregular shaped body.

The next interesting pictures arise when *adding* the effects of two masses, or two charges. It is the very nature of *vectors*, characterized by addition and scalar multiplication, which makes vector fields so useful to model force fields in physics. With modern computer tools it is very easy to graph e.g. the gravitational field in the neighborhood of two or three masses (which need not be equal).

**Exercise 1.7.1** *A* gravitational field in the plane *generated by a mass* $m_1$ *at the point* $(x_1, y_1)$ *is modeled (using suitable units such that the gravitational constant has unit magnitude* $G = 1$*)*

*by*

$$\vec{G}_1(x,y) = -m_1 \cdot \frac{(x-x_1)\vec{\imath} + (y-y_1)\vec{\jmath}}{(x-x_1)^2 + (y-y_1)^2} \tag{1.12}$$

*Plot the vector fields that results from superpositioning the gravitational fields of two or three different* point masses. *In the case of three masses consider generic configurations, as well as special ones like all three masses on a line, or at the corners of an equilateral triangle.*

Plotting hints: *Depending on the choice of your graphing tool, you may need to* truncate *the vector fields near the* singularities. *In general you may have to redefine your function $\vec{G}$ such that $\vec{G}(x,y) = 0$ whenever $(x-x_1)^2 + (y-y_1)^2 < \varepsilon^2$ for some* cutoff *value $\varepsilon$ that you choose. Other graphing programs may allow you to simply* pre-multiply *the above formula by a function that is zero if $(x-x_1)^2 + (y-y_1)^2 < \varepsilon^2$. Depending on what functions are predefined on your system, you may use the Heaviside function $u$, the signum function, or create a cutoff function yourself via $u(x) = (x + |x|)/2$.*

$$\vec{G}_1^*(x,y) = -m_1 \cdot u\left((x-x_1)^2 + (y-y_1)^2 - \varepsilon^2\right) \cdot \frac{(x-x_1)\vec{\imath} + (y-y_1)\vec{\jmath}}{(x-x_1)^2 + (y-y_1)^2} \tag{1.13}$$

Similar to velocity fields above, a natural question is to ask which path will small masses, small charges take under the influence of a gravitational, electric, or magnetic field $\vec{F}(x,y) = F_1(x,y)\vec{\imath} + F_2(x,y)\vec{\jmath}$. A first guess might again be the solution curves of the system of differential equations $\dot{x} = F_1(x,y)$, $\dot{y} = F_2(x,y)$. A little reflection makes clear that this can't be true: The moon, and many human-made satellites, continue to move along roughly circular orbits about the Earth. However, this motion is practically subject to only the Earth's gravitational field which points radially inwards towards the center of the Earth.
The solution is simple: Newton's third law (or rather *model*) says that force equals mass times acceleration. The Moon's and satellites' masses don't vary, and thus the force field is basically an *acceleration field*. Thus an appropriate system of differential equations is a second order system in the plane:

$$\begin{array}{rcl} \dot{x} &=& v_1 \\ \dot{y} &=& v_2 \\ \dot{v}_1 &=& \frac{1}{m} F_1(x,y) \\ \dot{v}_2 &=& \frac{1}{m} F_2(x,y). \end{array} \tag{1.14}$$

Here $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath}$ is the velocity vector of the test mass or test charge. As a second order system, one needs to provide the initial velocities as well as the initial positions. The more detailed study of this system of differential equations is usually part of a differential equations course, and beyond the scope of this introduction to the calculus of vector fields.

Nonetheless, in many scenarios the masses or charges are very small, and they travel very slowly, e.g. due to friction, air-resistance or the like. In such cases one may be able to neglect the effects of inertia and approximate the true paths by the solution curves of the first order system $\dot{x} = F_1(x,y)$, $\dot{y} = F_2(x,y)$. (This is tantamount to pretending that the velocities are multiples of the forces.) It is common practice to call the integral curves of this first order system the *flow-lines* of the vector field $\vec{F}$. But we need to keep in mind that in general, particles, masses, charges and the like *do not* travel along these flow lines. Nonetheless, we shall study these flow lines in their own right, as they provide additional insight into which force fields might be gradient fields.

A briefly look at the other integral suggested in our initial discussion of velocity fields, provides little insight at this time: What would we mean by how much force flows across a surface (or curve)? In applications there will indeed be a need for such flux integrals, but they are not the kind of questions suitable to motivate a first study of integrals of vector fields.

Instead *force fields* suggest a different question that is related to work and energy, and to gradient fields. If you throw a ball into the air, it starts with lots of kinetic energy (it is moving fast). The *climb against the gravitational field* of the Earth requires *work* (in the technical sense of modern physics). This work uses up the kinetic energy until the vertical velocity becomes zero, and the ball starts again falling down. The physicist says that as the ball climbed it gained *potential energy.* Practically this means that due to its height, the ball could *do work*, again accelerate and gain kinetic energy by giving up some of its potential energy, alas height.
In a slightly different scenario, the space shuttle climbs against the gravitational field, again *doing work* and gaining potential energy. This time it is the rocket engines that provide the thrust, which both accelerate and raise the shuttle.
The main concern for us is to develop a mathematical structure and formalism that is suitable to model the *work* being done, and which captures the concept of potential energy.
The starting point is the work $\Delta W$ that is done by moving in a constant vector field $\vec{F}$. If the total displacement is $\vec{\Delta s}$, then the work equals *by definition* the scalar product

$$\Delta W = \vec{F} \cdot \vec{\Delta s} \tag{1.15}$$

For example, if we throw a ball we may assume that the gravitational field is *practically* constant along the flight path. However, the space shuttle experiences a weakening gravitational field. This changing field, together with the curved path, calls for calculus. The key idea is simple, and familiar from any kind of integral calculus: Chop up the flight path into small segments so that along each segment the field is practically constant. The pieces must also be small enough so that each segment of the path is practically linear! Then use the simple formula (1.15) on each piece, and add up the results. If properly done, we expect that it will be possible to consider the limit as the segments become infinitely small, which in turn gives rise to a notion of line integral. Indeed, this line integral will be the same as the one desired in the preceding sections on gradient fields.

This raises an immediate question: Are all, or some force fields gradient fields? If yes, what is the physical meaning of *"the function that the field is the gradient of"*? Maybe not unexpectedly, if the force field is a gradient field – and some, but not all are – then this function will play the role of a *potential energy*, possibly best thought off as *stored work*, or *ability (potential!) to do work*. A major theme in the sequel is the in-depth study of the interconnections between line integrals, whether a field is a gradient field, and the potential function (if there is one).

A final look at the units of force fields is less revealing, but still useful. The standard unit for a force is one Newton, which is defined as one kilogram times one meter, divided by one second squared. This may sound complicated, but makes sense in light of Newton's third law. Integrating a force with respect to distance yields Newton meters, which are also known as Joules, the basic unit of energy. In electric settings the potential difference is measured in volts, and the units for the electric field are volts per meter. This appears more suggestive: Changing the units for distances in the domain from meters to centimeters will increase the coordinates of a point by a factor of 100. However, the numerical value of the electric field, now measured in volts per centimeter will decrease by a factor of 100. This suggests that electric fields be regarded as co-variant vector fields, further suggesting a close relation to gradient fields. This electric example also suggests that for general force fields one might consider the magnitude of the field in (Joules per meter) rather than in Newtons!

# Chapter 3

# Line integrals

## 3.1 Introduction

The preceding section suggested the need for a number of different notions of integrals of vector fields. Flow-lines, that is, integral curves of the associated system of differential equations are one kind of integral. They are traditionally studied in courses on differential equations. Our focus here is to develop two different kinds of line integrals. The development is guided by the desire that the line integrals are useful to provide solutions for the practical problems considered before: One notion of line integral shall provide the work done when traveling in a force field. One (as it turns out, the same) line integral shall recover the original *potential* function in the case that the vector field is a gradient field. Another line integral shall yield the *total flux* of a moving fluid or gas across a curve.

The first two applications work as well in three dimensions as they do in the plane. However, the flux is different in each case: In the plane we have the flux across a curve. In 3-space we consider the flux across a surface. However, we shall initially focus on vector fields and their integrals in the plane as these are much easier to visualize, and they already exhibit most of the features of interest.

Some key points that are worth to be pointed out before getting into the details.

- Every kind of integral assigns *a number* to a *pair of objects*. One, the *integrand*, is *"to be integrated"*. The other is the object *over which* the first is to be integrated.
  In the first calculus course, the pair consists of a (piecewise continuous) function $f$, and a finite interval $[a, b]$. The resulting number is denoted $\int_a^b f(u)\,du$.
  For line integrals, the pair consist of a vector field $\vec{F}$ and a curve $C$. One traditional notation for the resulting number uses the symbol $\int_c \vec{F} \cdot d\vec{R}$.
  Either notation is loaded with ambiguities. The first step towards more precise terminology is to consider differential forms $f\,dx$ or $\vec{F} \cdot d\vec{R}$ in place of the function $f$ or vector field $\vec{F}$. The detailed study of *differential forms* is subject of advanced courses.

- Every integral is a *limit of sums*. The key idea (in Riemann integrals) is to *break* the interval, or curve, or surface, or region in the plane, or ... into smaller and smaller pieces. The usual assumption that the integrand is (piecewise) continuous in a suitable sense means that it will be practically constant on each piece – provided the pieces are *small enough*. The precise technical hypotheses which make this intuitive foregoing rigorous are the subject of (mathematical) *analysis*.

- Every integral is related to some notion of derivative via some form of a *fundamental theorem*. The basic theme is that integration and differentiation are inverse operations in a suitable sense.
  Recall that an integral yields a *number* for every fixed interval $[a, b]$, curve $C$, surface $S$, $\ldots$, and any kind derivative of a number is zero. Thus it is clear that any such fundamental theorem must employ some subtle changes of point of view. In first year calculus the key for one fundamental theorem is to consider the integral $\int_a^b f(u)\,du$ as a function of the upper limit $b$ of integration. The two fundamental theorems state that $\frac{d}{db}\int_a^b f(u)\,du = f(b)$, and $f(b) - f(a) = \int_a^b f'(u)\,du$ for suitably nice functions $f$.
  Vector calculus has its own generalized fundamental theorems. The second fundamental theorem of first year calculus, Green's theorem, the divergence theorem and Stokes' theorem are all special cases of one general Stokes' theorem. A key to success is to understand the general structure common to all these special cases.

## 3.2 Line integrals

For a starter, consider the question about how much work is required to lift a satellite into orbit. For simplicity think of the satellite being *shot* into orbit, rather than being carried by a rocket. In practice most of the work is used to lift up the fuel that is burnt at a higher altitude. The space shuttle flies at a height of about 190 kilometers (about 120 miles), whereas geostationary satellites orbit about 35,800 kilometers (22,300 miles) above the surface and 42,000 kilometers from the center of the Earth.

The magnitude of the gravitational field of the Earth is inversely proportional to the square of the distance from the center of the Earth. Thus, on the typical orbits of the space shuttle and of geostationary satellites the magnitudes are about

$$\left(\frac{6,270}{6,270+190}\right)^2 \approx 94.2\% \quad \text{and} \quad \left(\frac{6,270}{6,270+35,800}\right)^2 \approx 2.22\% \tag{3.1}$$

of the magnitude at the surface, respectively. Thus one may obtain a very crude estimate for the space shuttle by assuming that the gravitational field is constant. However, for geostationary satellites one cannot avoid using integral calculus to account for the changing field. On the other hand, when calculating trajectories of base-balls and other objects in everyday surroundings we are used to assuming that the gravitational field is constant – both in magnitude and direction!

The key idea in the calculation of the work required to lift the satellite into orbit is to break up the flight path into many small pieces. These should be so small that on each piece one may reasonably assume that the gravitational field is constant. The second issue is the curved flight path. Again, if the path is broken up into enough small pieces one may reasonably approximate each piece by a straight line segment.

On each of these small pieces use the definition for work from physics

$$\Delta W = \vec{F} \cdot \vec{\Delta R} \tag{3.2}$$

where $\vec{F}$ is a constant force, and $\vec{\Delta R}$ is a displacement vector. Recall, that is $\vec{F}$ and $\vec{\Delta R}$ are parallel then $\Delta W = \pm|\vec{F}|\,|\vec{\Delta R}|$. On the other hand, if $\vec{F}$ and $\vec{\Delta R}$ are perpendicular to each other, then $\Delta W = 0$.

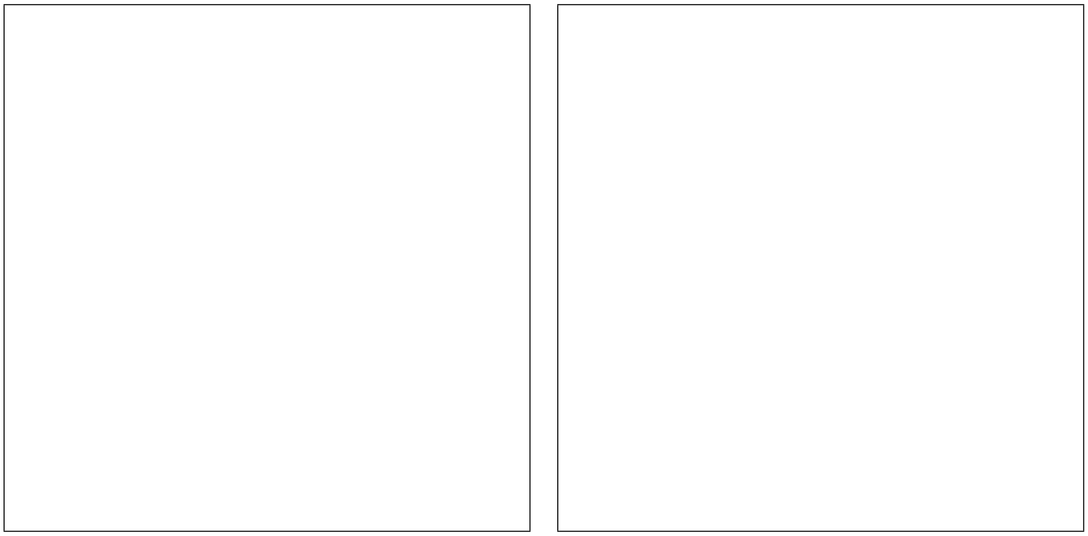


Figure 3.1: A path through a gravitational field on large and small scales

In general, suppose $P$ and $Q$ are the endpoints of a curve. Subdivide the curve into small segments with respective endpoints $P_i$, $i = 0, \ldots n$ with $P_0 = P$ and $P_n = Q$. Let $\vec{\Delta R_i} = \overrightarrow{P_{i-1}P_i}$ denote the respective displacement vectors between the endpoints of each segment. Write $\vec{F_i}$ for the value of $\vec{F}(x, y)$ at points $(x, y)$ on, or in the vicinity of, the $i$-th segment of the curve. Then an approximate value for the total work is

$$\Delta W = \sum_{i=1}^{n} \vec{F_i} \cdot \vec{\Delta R_i} \tag{3.3}$$

Consider a specific example. Let $\vec{F}(x, y) = \frac{-1}{y^2}\vec{\jmath}$ and let $C$ be the quarter circle with center $(-2, 2)$ from $(0, 2)$ to $(-2, 4)$. It is a good practice to make a quick estimate, aided by some graphs if necessary, before starting any detailed calculations. The total length of the curve is equal to $\pi \approx 3$, and the magnitude of the vector field varies between 0.25 when $y = 2$ and 0.0625 when $y = 4$. Thus the numerical value of the total work is expected to be considerably less than 1. Since the vector field points always *down*, and the path goes *up*, and thus the total work should be negative.

Choosing a small value for the number of pieces, create the following tables: The first columns contain the coordinates of the endpoints of the segments of the curve. The last column of the first table contains the values of the vector field at these *sample points*. The second table has one fewer row the the first, since the number of segments is always one less than the number of endpoints. The first column contains the displacement vectors between the endpoints of the segments. The middle column lists the average value of the vector field at the endpoints of each segment. The last column lists the dot products of the respective averaged values of the vector field and of the displacement vectors.

| $i$ | $x_i$ | $y_i$ | $\vec{F}(x_i, y_i)$ |
|---|---|---|---|
| 0 | 0.00 | 2.000 | $0\vec{\imath} - .2500\vec{\jmath}$ |
| 1 | -0.017 | 2.261 | $0\vec{\imath} - .1956\vec{\jmath}$ |
| 2 | -0.068 | 2.518 | $0\vec{\imath} - .1577\vec{\jmath}$ |
| 3 | -0.152 | 2.766 | $0\vec{\imath} - .1307\vec{\jmath}$ |
| 4 | -0.268 | 3.000 | $0\vec{\imath} - .1111\vec{\jmath}$ |
| 5 | -0.413 | 3.218 | $0\vec{\imath} - .0966\vec{\jmath}$ |
| 6 | -0.586 | 3.414 | $0\vec{\imath} - .0858\vec{\jmath}$ |
| 7 | -0.783 | 3.587 | $0\vec{\imath} - .0777\vec{\jmath}$ |
| 8 | -1.000 | 3.732 | $0\vec{\imath} - .0718\vec{\jmath}$ |
| 9 | -1.234 | 3.848 | $0\vec{\imath} - .0675\vec{\jmath}$ |
| 10 | -1.482 | 3.932 | $0\vec{\imath} - .0647\vec{\jmath}$ |
| 11 | -1.739 | 3.983 | $0\vec{\imath} - .0630\vec{\jmath}$ |
| 12 | -2.000 | 4.000 | $0\vec{\imath} - .0625\vec{\jmath}$ |

| $i$ | $\Delta\vec{R}_i$ | $\vec{F}_i$ | $(\Delta W)_i = \vec{F}_i \cdot \Delta\vec{R}_i$ |
|---|---|---|---|
| 1 | $-0.017\vec{\imath} + 0.261\vec{\jmath}$ | $-0.2228\vec{\jmath}$ | $-0.0582$ |
| 2 | $-0.051\vec{\imath} + 0.257\vec{\jmath}$ | $-0.1766\vec{\jmath}$ | $-0.0454$ |
| 3 | $-0.084\vec{\imath} + 0.248\vec{\jmath}$ | $-0.1442\vec{\jmath}$ | $-0.0358$ |
| 4 | $-0.116\vec{\imath} + 0.234\vec{\jmath}$ | $-0.1209\vec{\jmath}$ | $-0.0283$ |
| 5 | $-0.145\vec{\imath} + 0.218\vec{\jmath}$ | $-0.1038\vec{\jmath}$ | $-0.0226$ |
| 6 | $-0.173\vec{\imath} + 0.196\vec{\jmath}$ | $-0.0912\vec{\jmath}$ | $-0.0179$ |
| 7 | $-0.197\vec{\imath} + 0.173\vec{\jmath}$ | $-0.0818\vec{\jmath}$ | $-0.0141$ |
| 8 | $-0.215\vec{\imath} + 0.145\vec{\jmath}$ | $-0.0748\vec{\jmath}$ | $-0.0108$ |
| 9 | $-0.234\vec{\imath} + 0.116\vec{\jmath}$ | $-0.0697\vec{\jmath}$ | $-0.0081$ |
| 10 | $-0.248\vec{\imath} + 0.084\vec{\jmath}$ | $-0.0661\vec{\jmath}$ | $-0.0056$ |
| 11 | $-0.257\vec{\imath} + 0.051\vec{\jmath}$ | $-0.0639\vec{\jmath}$ | $-0.0033$ |
| 12 | $-0.261\vec{\imath} + 0.017\vec{\jmath}$ | $-0.0628\vec{\jmath}$ | $-0.0011$ |

Summing the products in the last columns yield the approximate value $W \approx -0.2511$. This value is close to the estimate, and appears reasonable. Using finer subdivisions, one expects that the values become more accurate, and that they approach a limit.

| n | 3 | 6 | 12 | 24 | 48 | 96 | 512 | 1024 |
|---|---|---|---|---|---|---|---|---|
| W | -0.2655 | -0.2542 | -0.2511 | -025036 | 0.250067 | -0.250016 | -0.250001 | -0.250000 |

**Exercise 3.2.1** *Repeat the calcualtions of the example,. but instead of using the adapted* trapezoidal rule, *use an adapted midpoint rule. That is, replace* $\vec{F}_i = \frac{1}{2}(\vec{F}(x_{i-1}, y_{i-1}) + \vec{F}(x_i, y_i))$, *by* $\vec{F}_i = \vec{F}(\frac{1}{2}(x_{i-1} + x_1), \frac{1}{2}(y_{i-1} + y_i))$.

**Exercise 3.2.2** *Use* $N = 12$ *segments to numerically estimate the line integral of the vector field* $\vec{M}(x, y) = (-y\vec{\imath} + x\vec{\jmath})/(x^2 + y^2)$ *over the unit circle* $x^2 + y^2 = 1$ *traversed counterclockwise.*

**Exercise 3.2.3** *Write a procedure in any software package that creates data as in the tables above, and uses these to obtain a numerical estimate of the line integral for any pair of a vector field and a curve. Test your procedure on the example worked above.*

finer subdivision
define line integral
convergence criteria
piecewise $C^1$ and $C^0$ uniform easy proof appendix
proposition estimate upper bound
calculate use parameterization
independent of parameterization
intuitively clear
proof simple chain rule

## 3.3 Conservative vector fields and the fundamental theorem

special case gradient field
fundamental theorem for line integrals 2d and 3d
closed curves zero

## 3.4 "Flux" line integrals"

incl incompressible

# Chapter 4

# Green's theorem

## 4.1 Introduction

Only very few vector fields are gradient fields. For most pairs of a vector field and a closed curve one may expect the line integral is different from zero. This chapter analyzes the *defect* of vector fields that might not be conservative. The final highlights are the formulation of Green's theorem, powerful demonstrations of typical uses, and compelling notions of derivatives of vector fields.

The development is deliberately slow and thorough. The rationale is that the case of vector fields in the plane is the most accessible, and the development of Green's theorem is a natural model for other versions of Stokes' theorem, in three and higher dimensions.

The emphasis is to develop a deep and coherent understanding of derivatives of vector fields, and of the associated analogues of the fundamental theorem of calculus. A main theme is that before one attempts to understand differential calculus and the fundamental theorem of calculus, one ought to first develop a profound understanding the linear setting. After all, differentiability merely means that on a small scale the object is well approximable by a linear object. Moreover, in the linear context any fundamental theorem reduces to a *precalculus* (or *calculus-free*) statement that everyone should comprehend first, including the basic ideas behind its proof.

The main story line for this chapter is the investigation of the implications of not being conservative. As this will eventually lead to developing notions of derivatives of vector fields, it is wise to prepare by formally introducing *linear vector fields.*

## 4.2 Linear vector fields

derivatives? infinitesimal linearity, should understand linear before taking derivatives also good examples of simple fields that are easy to explore
linearity and superposition
input: points versus vectors
linearity: formula easy, given just two vectors
matrix

$$\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath} \quad = \quad L_1(x,y)\,\vec{\imath} + L_2(x,y)\,\vec{\jmath}. \tag{4.1}$$

or using column vectors

$$\vec{L}(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.2}$$

Easy to find formula from picture!

$$\begin{array}{rclcl} a & = & \vec{\imath}\cdot\vec{L}(0,1) & = & L_1(1,0) \\ c & = & \vec{\jmath}\cdot\vec{L}(0,1) & = & L_2(1,0) \end{array} \qquad \begin{array}{rclcl} b & = & \vec{\imath}\cdot\vec{L}(1,0) & = & L_1(0,1) \\ d & = & \vec{\jmath}\cdot\vec{L}(1,0) & = & L_2(0,1) \end{array} \tag{4.3}$$

81 cases each of the coefficients $a, b, c, d$ taking one of the values $-1, 0, +1$.
Linear fields in polar coordinates, use $\vec{u}_r$ and $\vec{u}_\Theta$. Linearity defn as opposed to looks linear

## 4.3 Green's theorem for linear fields

Only very few vector fields are gradient fields. For most pairs of vector fields and closed curves one may expect the line integral to be different from zero. This section begins a closer investigation of such vector fields. This back-door approach will, quite surprisingly lead to the development of a notion of derivative for vector fields, and associated analogues of the fundamental theorem of calculus.

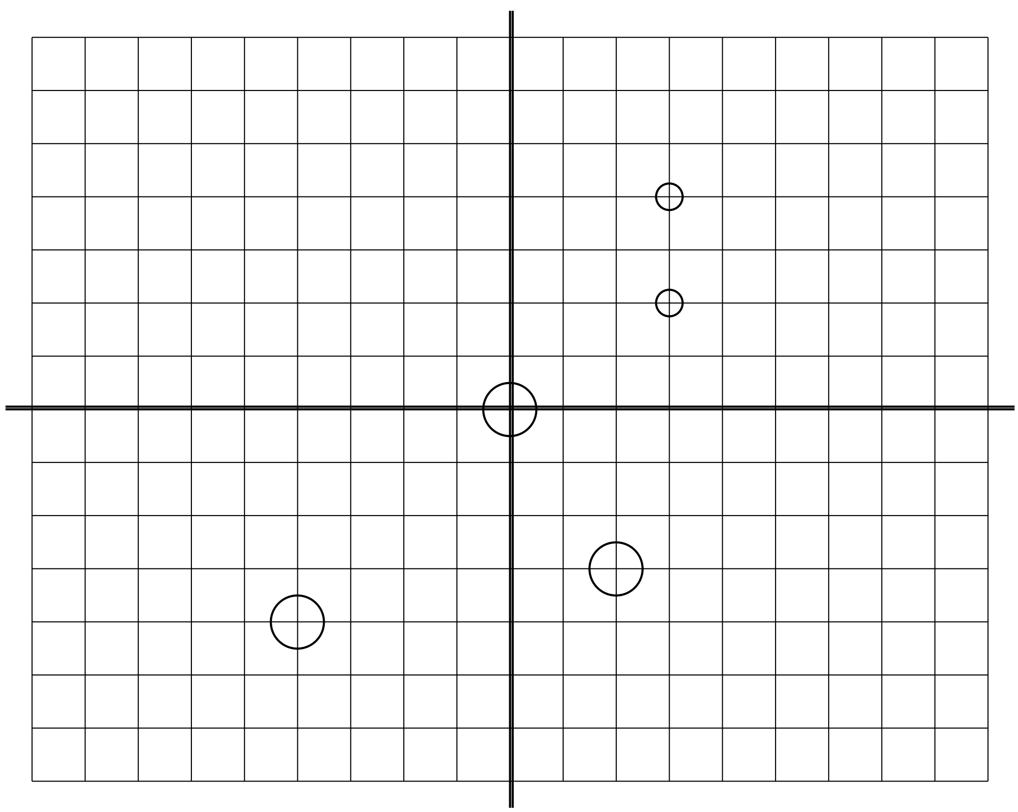
Rather than considering the most general vector fields and general curves, it makes sense to start analyzing the structurally most simple vector fields and curves. Note that every constant vector field is a gradient field. Hence the line integrals of any constant field over any closed curve is zero.

**Exercise 4.3.1** *For a constant vector field $\vec{F}(x,y) = a\vec{\imath} + b\vec{\jmath}$ find a potential function $\varphi(x,y)$.*

The structurally next most simple case is that of linear vector fields

$$\vec{L}(x,y) = (ax + by)\vec{\imath} + (cx + dy)\vec{\jmath} \tag{4.4}$$

**Exercise 4.3.2** (**Class-exercise**:)
*Consider the linear vector field $\vec{L}(x,y) = (3x - 2y)\vec{\imath} + (5x + 8y)\vec{\jmath}$.*
*(i) For each of the closed curves depicted below evaluate the line integral $\oint_C \vec{L} \cdot \vec{dr}$.*



*Add your own triangles, squares, rectangles, semi-and quarter circles, ...*

*and tabulate the results*

| *Student name* | *Contour* | *Line integral* | | |
|---|---|---|---|---|
| | *Circle* $C_1$ | 21.9911 | | |
| | *Square* $C_2$ | 28 | | |
| | *Triangle* $C_3$ | 3.5 | | |
| | | | | |
| | | | | |
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| | | | | |

*(ii) Look for patterns. Use the last two columns for additional data as needed. Make conjectures. Test the conjectures by evaluating additional line integrals over other curves, or by modifying the vector field.*

*Your conjecture should be powerful enough to predict – with only minimal calculation, without paper or calculator! – the value of any line integral of a linear vector field over any closed curve similar to those considered above.*

**DO NOT READ ON BEFORE COMPLETING THE EXERCISE ABOVE!**
**Work together. YOU CAN discover some deep math here!**
**Don't miss the opportunity.**

The preceding exercise should have led to some remarkable observations.

**Conjecture 4.3.1** *For a linear field the line integral is scaled by the area of the region $\mathcal{R}$ enclosed by the curve $C$: The ratio of the line integral divided by the area*

$$\frac{\oint_C \vec{L}\cdot d\vec{r}}{(\textit{area of } \mathcal{R})} = \textit{ constant.} \tag{4.5}$$

*is independent of the location, the shape, and the size of the curve $C$. The constant equals $(c-b)$ (in above notation) and only depends on the vector field.*

Compare this with the slope of a line: The difference quotient (*"rise over run"*)

$$\frac{y_2 - y_1}{x_2 - x_1} = m \;\text{ is constant.} \tag{4.6}$$

The constant depends only on the choice of the line, but is independent of both the location of the points, and the size of the increment $(x_2 - x_1)$. Of course, this constant is the slope of the line, and it plays a central role in the definition of the derivative of a function of a single variable. In analogy the quotient $(c - b)$ plays a key role in vector calculus, especially in the definition of the curl. It is appropriate to consider $(c - b)$ as a generalization of *"slope"* of a straight line.

The conjecture deserves proof – this special case will lead to Green's theorem! From there we will proceed in small steps, from simple regions to increasingly more general curves. In the next sections we continue with the generalization to nonlinear vector fields. In the end we shall have developed both a compelling new notion of a derivative for vector fields, and a powerful and useful analogue of the fundamental theorem of calculus.

A natural first step is to consider rectangles whose sides are aligned with the coordinate axes.
The analysis is much facilitated by the observation that for linear vector fields integrated over line segments the line integral really does not require any calculus at all: Both the midpoint and the trapezoidal rule are exact!

**Exercise 4.3.3** *Verify that for a linear vector field $\vec{L}$ and a line segment $C$ from $(x_1, y_1)$ to $(x_2, y_2)$ the line integral equals*

$$\begin{aligned}\oint_C \vec{L}\cdot d\vec{r} &= \vec{L}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\cdot\Big((x_2-x_1)\vec{\imath} + (y_2-y_1)\vec{\jmath}\Big) \\ &= \left(\frac{\vec{L}(x_1,y_1)+\vec{L}(x_2,y_2)}{2}\right)\cdot((x_2-x_1)\vec{\imath} + (y_2-y_1)\vec{\jmath})\end{aligned}$$

*(i) Parameterize the line segment and directly evaluate the line integral for the field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$.*

*(ii) Argue pictorially, using Riemann sums. Hint: Subdivide the line segment into a number of smaller segments of equal lengths. Group the vectors together analogous to the famous solution supposedly provided by Gauss as a school child when asked to sum* $1+2+3+...99+100 = 100+(1+99)+(2+98)+\ldots(49+51)+50 = 50*100+50 = 5050$.
*(iii) Argue geometrically using only the abstract characterization of linearity (i.e without referring to any coordinates) and the vector-form of the parameterization of a line segment* $\vec{r}(t) = \vec{r}_1 + t(\vec{r}_2 - \vec{r}_1)$, $t = 0..1$.

**Lemma 4.3.2** *The line integral of the linear vector field* $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ *over the edges of the rectangle with corners* $(x_0 \pm \Delta x, y_0 \pm \Delta y)$ *equals*

$$\oint_C \vec{L}\cdot d\vec{r} = (c-b)\cdot 4\Delta x\Delta y \qquad (4.7)$$

**Proof.** (outline). Use the observation about the exactness of the midpoint rule, and calculate:

$$\begin{aligned}
\oint_C \vec{L}\cdot d\vec{r} &= \oint_{C_1} \vec{L}\cdot d\vec{r} + \oint_{C_2} \vec{L}\cdot d\vec{r} + \oint_{C_3} \vec{L}\cdot d\vec{r} + \oint_{C_4} \vec{L}\cdot d\vec{r} \\
&= \vec{L}(x_0, y_0 - \Delta y)\cdot(2\Delta x\vec{\imath}) \quad + \quad \vec{L}(x_0+\Delta x, y_0)\cdot(2\Delta y\vec{\jmath}) + \\
&\quad \vec{L}(x_0, y_0 + \Delta y)\cdot(-2\Delta x\vec{\imath}) \quad + \quad \vec{L}(x_0-\Delta x, y_0)\cdot(-2\Delta y\vec{\jmath}) \\
&= (ax_0 + b(y_0 - \Delta y)\vec{\imath} + (\ldots)\vec{\jmath})\cdot(2\Delta x\vec{\imath}) \;+\; ((\ldots)\vec{\imath} + c(x_0+\Delta x) + dy_0\vec{\jmath})\cdot(2\Delta y\vec{\jmath}) + \\
&\quad (ax_0 + b(y_0 + \Delta y)\vec{\imath} + (\ldots)\vec{\jmath})\cdot(-2\Delta x\vec{\imath}) \;+\; ((\ldots)\vec{\imath} + c(x_0-\Delta x) + dy_0\vec{\jmath})\cdot(-2\Delta y\vec{\jmath}) \\
&= \ldots \\
&= (c-b)\cdot 4\Delta x\Delta y. \quad \blacksquare
\end{aligned}$$

Take a closer look how the scaling by area comes to be: It really consists of two components. On one side, there is the contribution of the increments $\Delta x$ and $\Delta y$ to the change in the vector field, e.g. comparing the vector field along the lower edge and along the upper edge. Dividing by the distance between these edges gives a measure of a *rate of change* of the vector field. The other increments $\Delta x$ and $\Delta y$ are contributed by the lengths of the curve segments. On a very small scale the vector field is almost constant, and the value of the line integral along each edge is almost directly proportional to the length of the line segment. Taken together, the line integral over the closed contour is scaled by the square of the linear dimension, or by the area of the region enclosed by the curve.

**Exercise 4.3.4** *Fill in the omitted details in the calculation above.*

**Exercise 4.3.5** *Replace the rectangle above by a triangle with two sides aligned with the coordinate axes. E.g. choose the corners to be* $(x_0, y_0)$, $(x_0+\Delta x, y_0)$, *and* $(x_0, y_0+\Delta y)$. *Carry out the analogous calculation.*

**Lemma 4.3.3** *Suppose $C$ is the curve consisting of the edges of the triangle with corners at $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ (oriented counter clockwise). If $\vec{L}$ is the linear vector field $\vec{L}(x, y) = (ax + by)\vec{\imath} + (cx + dy)\vec{\jmath}$, then*

$$\oint_C \vec{L} \cdot \vec{dr} = (c - b) \cdot (\text{ area of the triangle }) \tag{4.8}$$

**Exercise 4.3.6** *Prove the lemma* (4.3.3) *via direct calculation analogous to that for lemma* (4.3.2)

**Exercise 4.3.7** *Using linearity rewrite the vector field $\vec{L}$ as $\vec{L} = \vec{L}_0 + \vec{\Delta L}$ where $\vec{L}_0(x, y) = \vec{L}(x_0, y_0)$ for all $(x, y)$ and $\vec{\Delta L}(x, y) = \vec{L}(x - x_0, y - y_0)$.*
*(i) Explain why $\oint_C \vec{L}_0 \cdot \vec{dr} = 0$ and hence $\oint_C \vec{L} \cdot \vec{dr} = \oint_C \vec{\Delta L} \cdot \vec{dr}$.*
*(ii) Evaluate $\overrightarrow{\Delta L}$ at the midpoints of the four edges of the rectangle. Use the midpoint rule to evaluate $\oint_C \vec{\Delta L} \cdot \vec{dr}$.*
*(iii) Alternatively, observe that $\overrightarrow{\Delta L}\,(x_0 + \Delta x, y_0 + \Delta y) = \vec{L}(\Delta x, \Delta y)$, and hence the line integral $\oint_C \vec{\Delta L} \cdot \vec{dr}$ equals $\oint_{C_0} \vec{L} \cdot \vec{dr}$ where $C_0$ is the curve $C$ translated back to the origin (i.e. each point $(x, y)$ is moved to the point $(x - x_0, y - y_0)$). Thus it suffices to consider rectangles centered at the origin! Repeat the calculations for the case of $(x_0, y_0) = (0, 0)$.*
*(iv) Compare these approaches with the calculations above. Which one illustrates best what is going on?*
*(v) Go even a step further. Using linearity to* combine *some of the terms occurring in the calculation. E.g., simplify $\vec{L}(\Delta x, 0) - \vec{L}(-\Delta x, 0) = 2\Delta x \vec{L}(1, 0)$, and eventually arrive at*

$$\oint_C \vec{L} \cdot \vec{dr} = \left(\vec{L}(1, 0) \cdot \vec{\jmath} - \vec{L}(0, 1) \cdot \vec{\imath}\right) \cdot (\text{ area of the region } \mathcal{R}). \tag{4.9}$$

*Note that this formulation does not make any explicit reference to the coefficients of the vector field when written out in rectangular coordinates. This very geometric point of view is most useful when working with polar or other* curvilinear *coordinates.*

So far we have proven that the conjecture holds true for (linear vector fields integrated over) rectangles that are aligned with the coordinate axes, and, in the exercises, for any triangles. The next step is to establish that the conjecture is true for (linear vector fields integrated over) arbitrary polygonally bounded regions.

It is intuitively clear, but takes a little work to prove rigorously that every region in the plane that is bounded by a polygonal curve, i.e. made up of line segments that do not intersect, can be *triangulated*. This means that the region can be decomposed into a union of a finite number of triangles, any two of which have either no points in common, one corner in common, or one edge in common.

**Exercise 4.3.8** *Develop a general argument why any region that is bounded by a polygonal curve can be triangulated as described above. Note, this does not require any calculus. A rigorous typically will require an* induction argument.

Consider a polygonal curve $C$ that encloses a region $\mathcal{R}$ as in the illustration. Suppose that the region is decomposed into a union of triangles (and/or rectangles) labeled $\mathcal{R}_1, \mathcal{R}_2, \ldots \mathcal{R}_n$ (which only meet on their corners or along their edges). Denote the *boundaries* of the regions $\mathcal{R}_k$ by $\mathcal{C}_k = \partial\mathcal{R}_k$, respectively. It is important that all these curves are oriented in a compatible way, i.e. all are oriented counterclockwise.

The area of the entire region $\mathcal{R}$ is just the sum of the areas of the small regions $\mathcal{R}_k$. More exciting is that the sum of the line integrals $\oint_{C_k} \vec{L} \cdot \vec{dr}$ over all the edges of the small regions adds up to just the line integral $\oint_C \vec{L} \cdot \vec{dr}$ over the outside curve: The integrals over all the interior edges cancel as each is traversed once in either direction. Formally this reads:

$$\begin{aligned}
\oint_C \vec{L} \cdot \vec{dr} &= \sum_{k=1}^{n} \oint_{C_k} \vec{L} \cdot \vec{dr} \\
&= \sum_{k=1}^{n} (c-b) \cdot (\text{ area of } \mathcal{R}_k) \\
&= (c-b) \cdot \sum_{k=1}^{n} (\text{ area of } \mathcal{R}_k) \\
&= (c-b) \cdot (\text{ area of } \mathcal{R})
\end{aligned}$$

This proves the following intermediate result which deserves to be formulated as a lemma.

**Lemma 4.3.4** *If $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ $C$ is a polygonal curve then*

$$\oint_C \vec{L} \cdot \vec{dr} = (c-b) \cdot (\textit{ area of region inside } C\,) \tag{4.10}$$

**Exercise 4.3.9** *Consider a region $\mathcal{R}$ that lies between two polygonal curves $C_1$ and $C_2$, oriented as shown (i.e. so that the region always lies to the left as one moves along the curve). By carefully going through the steps of the previous arguments, show that it is still true that $\oint_C \vec{L} \cdot \vec{dr} = (c-b)(\textit{ area of } \mathcal{R})$ if $C$ now denotes the* generalized curve *consisting of the two* connected *pieces $C_1$ and $C_2$.*

**Exercise 4.3.10** *Develop a formula for the area of a region that is bounded by a polygonal curve. The formula should take as input the list $(x_1,y_1),\ldots,(x_n,y_n)$ of coordinates of the vertices (corners). Advice: Pick a suitable linear vector field $\vec{L}$ – there are many possibilities – so that $\oint_C \vec{L} \cdot \vec{dr} = 1 \cdot (\textit{ area of } \mathcal{R})$. Utilize the midpoint or trapezoidal rule, each of which is gives the exact value for linear vector fields.*

Finally consider regions that are bounded by piecewise smooth curves $C$. Any such curve may be arbitrarily closely approximated by a polygonal curve $C_P$. For the sake of clarity we may assume that the polygonal curve $C_P$ lies entirely *inside* the smooth curve $C$.
On one side we want the areas of the regions inside the curves $C$ and $C_P$ to be arbitrarily close together.
On the other hand we also want also the line integrals $\oint_C \vec{L} \cdot \vec{dr}$ and $\oint_{C_P} \vec{L} \cdot \vec{dr}$ to be arbitrarily close together. This requires two arguments.
The easy part is that the vector field $\vec{L}$ has almost identical values on *corresponding points* on the respective curves (by hypothesis it is uniformly continuous). It takes a little bit more work to justify that the polygonal curve can also be chosen such that also the *velocity vectors* $\frac{\vec{dr}}{dt}$ are arbitrarily close at *corresponding* points. [[*this deserves an elegant argument – idea is to zoom in very much, so that the smooth curve looks practically straight. The rest is just book-keeping, using uniformity, bounds on* $\vec{r}''$ ...]]. This completes the outline of the proof of the conjecture, formulated now as a proposition.

**Proposition 4.3.5** *Suppose $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$, and $R$ is a region in the plane (possibly with* "holes"*. Let $\partial R = C$ denote the oriented, piecewise smooth boundary of $R$. Then*

$$\oint_{\partial R} \vec{L} \cdot \vec{dr} = (c-b) \cdot (\text{ area of } R) \tag{4.11}$$

In words: For line integrals of linear vector fields over any region that is bounded by piecewise smooth curves: The ratio of the line integral divided by the area of the enclosed region is a constant. This constant is independent of the location, the shape, and the size of the curve. The constant may be considered as an analogue of the slope of a straight line. As such it will be the precursor for a geometric definition of the *scalar curl*, one derivative of vector fields.
This proposition also reaffirms the curl test of the previous section in the special case of linear vector fields.

**Corollary 4.3.6** *A linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ is conservative if and only if $b = c$.*

### 4.3.1 Flux line integrals of linear fields

Practically every step of the previous section has an immediate analogue for flux line integrals in the plane. The following outline, in the form of a list of guided exercises, makes a great *project*. By going once more through the all steps, the overall argumentation, and the role of linearity will become even clearer.

**Proposition 4.3.7** *Suppose $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$, and $\mathcal{R}$ is a region in the plane (possibly with* "holes"*. Let $\partial\mathcal{R} = C$ denote the oriented, piecewise smooth boundary of $\mathcal{R}$. Then*

$$\oint_{\partial \mathcal{R}} \vec{L} \cdot \vec{dr} = (c-b) \cdot (\text{ area of } \mathcal{R}) \tag{4.12}$$

In words: For line integrals of linear vector fields over any region that is bounded by piecewise smooth curves: The ratio of the line integral divided by the area of the enclosed region is a constant. This constant is independent of the location, the shape, and the size of the curve. The constant may be considered as an analogue of the slope of a straight line. As such it will be the precursor for a geometric definition of the *scalar curl*, one derivative of vector fields.

**Exercise 4.3.11** *Repeat the class exercise* (4.3.2) *for flux line integrals* $\oint_C \vec{L}\cdot\vec{N}\,ds$. *Again start with the specific example* $\vec{L}(x,y) = (3x-2y)\vec{\imath} + (5x+8y)\vec{\jmath}$, *and evaluate the integral over several curves. Guided by your findings in exercise* (4.3.2), *experiment with somewhat different vector fields – e.g. experiment who a change of one of the coefficients affects the value of the integral. The final objective should be a carefully stated conjecture.*

**Exercise 4.3.12** *In analogy to exercise 4.3.3, verify that the trapezoidal and midpoint rules are exact for flux line integrals of linear vector fields* $\vec{L}$ *over any line segment* $C$ *from* $(x_1, y_1)$ *to* $(x_2, y_2)$. *I.e. verify (i) by direct calculation using a parameterization, (ii) pictorially as in exercise 4.3.3, or (iii) directly from the definition of linearity.*

$$\begin{aligned}\oint_C \vec{L}\cdot\vec{N}\,ds &= \vec{L}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\cdot\Big(-(y_2-y_1)\vec{\imath} + (x_2-x_1)\vec{\jmath}\Big) \\ &= \left(\frac{\vec{L}(x_1,y_1)+\vec{L}(x_2,y_2)}{2}\right)\cdot(-(y_2-y_1)\vec{\imath} + (x_2-x_1)\vec{\jmath}\vec{\jmath})\end{aligned}$$

**Lemma 4.3.8** *The flux line integral of the linear vector field* $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ *over the edges of the rectangle with corners* $(x_0 \pm \Delta x, y_0 \pm \Delta y)$ *equals*

$$\oint_C \vec{L}\cdot\vec{N}\,ds = (a+d)\cdot\,\Delta\,\Delta y \tag{4.13}$$

**Exercise 4.3.13** *Prove lemma* (4.3.8), *following the outline of the calculation in the proof of lemma* (4.3.2) *and exercise* (4.3.4).

**Exercise 4.3.14** *Carry out an analogous calculation for the case when the rectangle is replaced by a triangle with two sides aligned with the coordinate axes. E.g. choose the corners to be* $(x_0, y_0)$, $(x_0+\Delta x, y_0)$, *and* $(x_0, y_0+\Delta y)$.

**Lemma 4.3.9** *Suppose* $C$ *is the curve consisting of the edges of the triangle with corners at* $(x_1, y_1)$, $(x_2, y_2)$, *and* $(x_3, y_3)$ *(oriented counter clockwise). If* $\vec{L}$ *is the linear vector field* $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$, *then*

$$\oint_C \vec{L}\cdot\vec{N}\,ds = (a+d)\cdot(\text{ area of the triangle }) \tag{4.14}$$

**Exercise 4.3.15** *Prove the lemma* (4.3.9) *via direct calculation analogous to that for lemma* (4.3.8)

**Exercise 4.3.16** *In analogy to exercise* (4.3.7) *use linearity rewrite the vector field* $\vec{L}$ *as* $\vec{L} = \vec{L}_0 + \vec{\Delta L}$ *where* $\vec{L}_0(x,y) = \vec{L}(x_0,y_0)$ *for all* $(x,y)$ *and* $\vec{\Delta L}(x,y) = \vec{L}(x-x_0, y-y_0)$.
*(i) Explain why* $\oint_C \vec{L}_0\cdot\vec{N}\,ds = 0$ *and hence* $\oint_C \vec{L}\cdot\vec{N}\,ds = \oint_C \vec{\Delta L}\cdot\vec{N}\,ds$.
*(ii) Evaluate* $\overrightarrow{\Delta L}$ *at the midpoints of the four edges of the rectangle. Use the midpoint rule to*

*evaluate* $\oint_C \vec{\Delta L}\cdot\vec{N}\,ds$.

*(iii) Alternatively, observe that* $\overrightarrow{\Delta L}\,(x_0+\Delta x, y_0+\Delta y) = \vec{L}(\Delta x,\Delta y)$, *and hence the flux line integral* $\oint_C \vec{\Delta L}\cdot\vec{N}\,ds$ *equals* $\oint_{C_0}\vec{L}\cdot\vec{N}\,ds$ *where* $C_0$ *is the curve* $C$ *translated back to the origin (i.e. each point* $(x,y)$ *is moved to the point* $(x-x_0, y-y_0)$*). Thus it suffices to consider rectangles centered at the origin! Repeat the calculations for the case of* $(x_0,y_0)=(0,0)$.
*(iv) Compare these approaches with the calculations in the two preceding exercises. Which one illustrates best what is going on?*
*(v) Go even a step further. Using linearity to* combine *some of the terms occurring in the calculation. E.g., simplify* $\vec{L}(\Delta x,0)-\vec{L}(-\Delta x,0) = 2\Delta x\vec{L}(1,0)$, *and eventually arrive at*

$$\oint_C \vec{L}\cdot\vec{N}\,ds = \left(\vec{\imath}\cdot\vec{L}(1,0)+\vec{\jmath}\cdot\vec{L}(0,1)\right)\cdot(\text{ area of }\mathcal{R})\,. \tag{4.15}$$

**Exercise 4.3.17** *Similar to exercise* (4.3.17) *consider a region* $\mathcal{R}$ *that lies between two polygonal curves* $C_1$ *and* $C_2$*, oriented as shown in exercise* (4.3.17)*. The region always lies to the left as one moves along the curve, and thus the normal vector always points outward from the region. Note that for the inner curve this means that the outward normal points into the hole. By carefully going through the steps of prior arguments, show that it is still true that* $\oint_C \vec{L}\cdot d\vec{r} = (c-b)(\text{ area of }\mathcal{R})$ *if* $C$ *now denotes the* generalized curve *consisting of the two* connected *pieces* $C_1$ *and* $C_2$.

**Exercise 4.3.18** *Develop a formula for the area of a region that is bounded by a polygonal curve. The formula should take as input the list* $(x_1,y_1),\ldots,(x_n,y_n)$ *of coordinates of the vertices (corners). Advice: Pick a suitable linear vector field* $\vec{L}$ *– there are many possibilities – so that* $\oint_C \vec{L}\cdot d\vec{r} = 1\cdot(\text{ area of }\mathcal{R})$. *Utilize the midpoint or trapezoidal rule, each of which is gives the exact value for linear vector fields.*

**Proposition 4.3.10** *Suppose* $\vec{L}$ *is the linear vector field* $\vec{L}(x,y)=(ax+by)\vec{\imath}+(cx+dy)\vec{\jmath}$*, and* $R$ *is a region in the plane (possibly with* "holes"*. Let* $\partial R = C$ *denote the oriented, piecewise smooth boundary of* $R$*. Then*

$$\oint_{\partial R}\vec{L}\cdot\vec{N}\,ds = (a+d)\cdot(\text{ area of } R\,) \tag{4.16}$$

In words: For flux line integrals of linear vector fields over any region that is bounded by piecewise smooth curves: The ratio of the line integral divided by the area of the enclosed region is a constant. This constant is independent of the location, the shape, and the size of the curve. The constant may be considered as an analogue of the slope of a straight line. As such it will be the precursor for a geometric definition of the *divergence*, another derivative of vector fields.

## 4.4 Zooming for derivatives of vector fields

[[*This repeats the development of zooming for the Jacobian of a vector field in the DE section. Need to repeat it anyhow.*]]

Any notion of derivative is *inextricably* linked to approximability by linear objects (on a small scale). In the case of a function $f(x)$ of a single variable, this means pictorially that after sufficient zooming the graph is arbitrarily close to a straight line. The derivative $f'(a)$ at the point $a$ is defined as the slope of this line.

Zooming on the graph of a function $f(x)$

In order to *see* the derivative via zooming it is important that the zoom rates in domain and range are equal. In the case of a graph of a function $f(x)$ in single-variable calculus this means that the horizontal and vertical magnification rates must be equal.

In the case of a vector field the analogues of *horizontal* and *vertical* need clarification. The horizontal axis represents the domain, and is to be replaced by the $xy$-plane. The vertical axis represents the range – which is now represented by *arrows*, instead of *heights* ($y$-coordinates of points on the graph). We need to zoom at equal rates in the domain, here the $xy$-plane in the background, and in the range, here represented by arrows. One key observation is that when zooming for the derivative of $f$ at a point $(a, f(a)$ it is impossible to keep the "$x$-axis" on the screen! The vector field analogue means that it should be impossible to keep the *feet* of the arrows in view. We need to zoom in on the *arrow heads*.

A typical field on a small scale

The drift, a constant field

Compare the field and drift

Subtract the drift from the field

Viewing only the local changes

A simple way to obtain the desired pictures is to consider differences, analogous to the numerator $f(a + \Delta x) - f(a)$ in the difference quotient in single variable calculus. Fix a point $p = (x_0, y_0)$ where we want to zoom for the derivative of the vector field. Then compare the values $\vec{F}(x, y) = \vec{F}(x_0 + \Delta x, y_0 + \Delta y)$ of the vector field at points $(x, y) = (x_0 + \Delta x, y_0 + \Delta y)$ near $p$ to the value $\vec{F}(x_0, y_0$ at $p$. Define a constant vector field $F_0$ by $\vec{F}_0(x, y) = \vec{F}(x_0, y_0)$, and compare its values with those of the original field. Practically consider, and plot, the vector field

$$\overrightarrow{\Delta F}\ (x, y) = \vec{F}(x, y) - \vec{F}(x_0, y_0) \tag{4.17}$$

[[*Aside ([Move to appendix, include the picture!]): In the case of curved spaces it is not at all clear what one means by a constant vector field. Indeed, this question immediately leads to challenging questions in differential geometry, which have important consequences. For example according to Einstein's theory of general relativity the universe in which we live is curved according to the mass-density.*

**Exercise 4.4.1** *Use the* JAVA vector field analyzer*, or any other graphing tool to plot the vector field* $\overrightarrow{\Delta F}\ (x,y)$ *for* $(x,y)$ *very close to* $(x_0,y_0)$ *for the vector fields vecF and points* $p=(x_0,y_0)$ *listed below. In each case plot the field on smaller and smaller scales (i.e. for* $(x,y)=(x_0+\Delta x, y_0+\Delta y)$ *closer and closer to* $(x_0,y_0)$*. In each case write down the window size, i.e. basically the magnification, when the plotted vector fields stabilize, do not change any more upon further magnification.*

*a.* $\vec{F}(x,y)=\sin x\cdot\cos y\,\vec{\imath}+(\sin x+\cos y)\,\vec{\jmath}$*,* $p=(2,1)$.

*b.* $\vec{M}(x,y)=(-y\,\vec{\imath}+x\,\vec{\jmath})(/x^2+y^2)$*,* $p=(2,1)$.

*c.* $\vec{G}(x,y)=(-x\,\vec{\imath}-y\,\vec{\jmath})(/x^2+y^2)$*,* $p=(2,1)$.

*d.* $\vec{L}(x,y)=(2x-3y),\vec{\imath}+(3x+y)\,\vec{\imath}$*,* $p=(2,1)$.
*Explain how the plot relates to the statement that* the derivative of a linear function is the same linear function. *Compare to zooming in one the graph of* "linear" *[?] functions* $f\colon x\mapsto m\,x+b$ *and* $f\colon(x,y)\mapsto m\,x+n\,y+c$.

*Repeat the zooming at two other points of your choice.*

Note that most any plotting program will automatically scale the arrows so that they are of maximal length, but do not overlap. This basically automatically takes care of the analogue of the scaling factor $\frac{1}{h}$ in the first year calculus difference quotient $\frac{1}{h}(f(a+h)-f(a))$.
Due to the limited, very small, number of pixels available to draw each arrow, it takes surprisingly little magnification for the zooming to stabilize: Pixel for pixel the magnified plots of the vector field $\overrightarrow{\Delta F}$ show the limit ... In each case you should have observed that the final plot shows a linear vector field, a candidate suitable to be considered the derivative $\overrightarrow{DF}$ of the vector field $\vec{F}$ at the point $p$.
Before formalizing this notion, it is worthwhile to take a brief look at some numerical tables. The following table lists the values for the vector field $\overrightarrow{\Delta F}\ (x,y)=\vec{F}(x,y)-\vec{F}(x_0,y_0)$ when $\vec{F}(x,y)=\sin x\cdot\cos y\,\vec{\imath}+(\sin x+\cos y)\,\vec{\jmath}$ near the point $(x_0,y_0)=(2,1)$.

| $100\ \overrightarrow{\Delta F}$ | $x=1.98$ | $x=1.99$ | $x=2.00$ | $x=2.01$ | $x=2.02$ |
|---|---|---|---|---|---|
| $y=1.02$ | $-1.114\vec{\imath}-0.8796\vec{\jmath}$ | $-1.325\vec{\imath}-1.282\vec{\jmath}$ | $-1.540\vec{\imath}-1.694\vec{\jmath}$ | $-1.760\vec{\imath}-2.114\vec{\jmath}$ | $-1.985\vec{\imath}-2.544\vec{\jmath}$ |
| $y=1.01$ | $-0.335\vec{\imath}-0.030\vec{\jmath}$ | $-0.549\vec{\imath}-0.433\vec{\jmath}$ | $-0.768\vec{\imath}-0.844\vec{\jmath}$ | $-0.991\vec{\imath}-1.265\vec{\jmath}$ | $-1.220\vec{\imath}-1.695\vec{\jmath}$ |
| $y=1.00$ | $0.440\vec{\imath}+0.814\vec{\jmath}$ | $0.222\vec{\imath}+0.412\vec{\jmath}$ | $0.000\vec{\imath}+0.000\vec{\jmath}$ | $-0.227\vec{\imath}-0.421\vec{\jmath}$ | $-0.459\vec{\imath}-0.850\vec{\jmath}$ |
| $y=0.99$ | $1.209\vec{\imath}+1.653\vec{\jmath}$ | $0.989\vec{\imath}+1.250\vec{\jmath}$ | $0.763\vec{\imath}+0.839\vec{\jmath}$ | $0.532\vec{\imath}+0.418\vec{\jmath}$ | $0.296\vec{\imath}-0.012\vec{\jmath}$ |
| $y=0.98$ | $1.974\vec{\imath}+2.486\vec{\jmath}$ | $1.750\vec{\imath}+2.084\vec{\jmath}$ | $1.520\vec{\imath}+1.672\vec{\jmath}$ | $1.286\vec{\imath}+1.251\vec{\jmath}$ | $1.047\vec{\imath}+0.822\vec{\jmath}$ |

The table shows that $\overrightarrow{\Delta F}\ (x,y)$ while not linear is very close to linear. For example,

$$\begin{aligned}
\overrightarrow{\Delta F}\ (2.02,1.00) &= -0.459\,\vec{\imath}-0.850\,\vec{\jmath} \approx 2\cdot(-0.227\,\vec{\imath}-0.421\,\vec{\jmath}) = 2\cdot\overrightarrow{\Delta F}\ (2.01,1.00), \quad \text{and}\\
\overrightarrow{\Delta F}\ (1.99,1.02) &= -1.325\,\vec{\imath}-1.282\,\vec{\jmath} \approx (0.222\,\vec{\imath}+0.412\,\vec{\jmath})+(-1.540\,\vec{\imath}-1.694\,\vec{\jmath})\\
&= \overrightarrow{\Delta F}\ (1.99,1.00)+\overrightarrow{\Delta F}\ (2.00,1.02).
\end{aligned} \tag{4.18}$$

The table suggests a minor change in notation as every linear vector field $\vec{L}$ is zero at $(0,0)$. Here the role of $(0,0)$ is taken by $(x_0, y_0)$. Thus, instead of considering the difference $\overrightarrow{\Delta F}\ (x,y) = \vec{F}(x,y) - \vec{F}(x_0,y_0)$ as a function of $(x,y)$ near $x_0, y_0)$, it appears more suitable to consider the vector field

$$\overrightarrow{\Delta' F}\ (\Delta x, \Delta y) = \vec{F}(x_0 + \Delta x, y_0 \Delta y) - \vec{F}(x_0, y_0) \tag{4.19}$$

as a function of $\delta x, \Delta y)$ near $(0,0)$. the values inside the table remain unchanged, only the labels along the edges should be changed to $-0.02$, $-0.1$, $0.00$, $0.01$, and $0.02$. Now it is appropriate to say that $\overrightarrow{\Delta' F}$ is very close to being a linear vector field as for small values of $(\Delta x)_i$, $(\Delta y)_i$ and any constant $c$.

$$\begin{aligned} \overrightarrow{\Delta' F}\ (c \cdot \Delta_x, c \cdot \Delta y) &\approx c \cdot \overrightarrow{\Delta' F}\ (\Delta_x, \Delta y), \quad \text{and} \\ \overrightarrow{\Delta' F}\ ((\Delta x)_1 + (\Delta x)_2, (\Delta y)_1 + (\Delta y)_2) &\approx \overrightarrow{\Delta' F}\ ((\Delta x)_1, (\Delta y)_1) + \overrightarrow{\Delta' F}\ ((\Delta x)_2, (\Delta y)_2) \end{aligned} \tag{4.20}$$

Returning to the values of the table, it is easy to obtain an explicit formula for a linear vector field $\vec{L}$ that is a good approximation for $\overrightarrow{\Delta' F}$

$$\vec{L}(\Delta x, \Delta y) = (\,-0.227\Delta x - 0.786\delta y\,)\,\vec{\imath}(\,-0.421\Delta x - 0.844\Delta y\,)\,\vec{\jmath}. \tag{4.21}$$

The coefficients of this field may be obtained either from the values of $\overrightarrow{\Delta F}$, or even directly from $\vec{F}$ due to the identity

$$\overrightarrow{\Delta F}\ (x_0 + \Delta x, y_0 + \Delta y) - \underbrace{\overrightarrow{\Delta F}\ (x_0 + \Delta x, y_0 + \Delta y)}_{=0} = \vec{F}(x_0 + \Delta x, y_0 + \Delta y) - \vec{F}(x_0, y_0) \tag{4.22}$$

These differences may be considered as numerical approximations of the values of the partial derivatives of the components $\vec{\imath} \cdot \vec{F}(x,y) = F_1(x,y) = \sin x \cdot \cos y$ and $\vec{\jmath} \cdot \vec{F}(x,y) = F_2(x,y) \sin x + \cos y$ at $(x_0, y_0)$. The following table shows the approximate values obtained from the table, and the decimal approximations obtained by symbolically differentiating the components of $F$ and evaluating at $(x_0, y_0)$

$$\begin{aligned} a = \vec{\imath} \cdot \vec{L}(1,0) &\overset{!}{=} -0.227 \approx \tfrac{1}{0.01}\,\vec{\imath} \cdot\ (\vec{F}(x_0 + 0.01, y_0) - \vec{F}(x_0, y_0)) \approx \tfrac{\partial}{\partial x}\,\vec{\imath} \cdot \vec{F}(x_0, y_0) \approx -0.2248 \\ b = \vec{\imath} \cdot \vec{L}(1,0) &\overset{!}{=} -0.786 \approx \tfrac{1}{0.01}\,\vec{\imath} \cdot\ (\vec{F}(x_0, y_0 + 0.01) - \vec{F}(x_0, y_0)) \approx \tfrac{\partial}{\partial y}\,\vec{\imath} \cdot \vec{F}(x_0, y_0) \approx -0.4161 \\ c = \vec{\jmath} \cdot \vec{L}(1,0) &\overset{!}{=} -0.421 \approx \tfrac{1}{0.01}\,\vec{\jmath} \cdot\ (\vec{F}(x_0 + 0.01, y_0) - \vec{F}(x_0, y_0)) \approx \tfrac{\partial}{\partial x}\,\vec{\jmath} \cdot \vec{F}(x_0, y_0) \approx -0.7652 \\ d = \vec{\jmath} \cdot \vec{L}(1,0) &\overset{!}{=} -0.844 \approx \tfrac{1}{0.01}\,\vec{\jmath} \cdot\ (\vec{F}(x_0, y_0 + 0.01) - \vec{F}(x_0, y_0)) \approx \tfrac{\partial}{\partial y}\,\vec{\jmath} \cdot \vec{F}(x_0, y_0) \approx -0.8415 \end{aligned} \tag{4.23}$$

**Exercise 4.4.2** *Prepare a similar table of values for smaller values of* $(\Delta x, \Delta y)$*, e.g. for* $x = 1.998,\ 1.999, 2.000,\ 2.001,\ 2.002$ *and* $y = 0.998,\ 0.999, 1.000,\ 1.001,\ 1.002$. *Again check at a few values how close to* linear *the table is. Use the table to read off better numerical estimates for the partial derivatives.*

**Exercise 4.4.3** *Prepare a similar tables of values for the vector field* $\vec{M}(x,y) = (-y\vec{\imath} + x\vec{\jmath})/(x^2 + y^2)$. *Again check at a few values how close to* linear *the table is. Use the table to read off better numerical estimates for the partial derivatives. In particular, what are the values of* $(c - b)$ *and* $(a + d)$ *in your approximation, and what is the significance of these values?*

**Exercise 4.4.4** *Repeat the previous exercise for* $\vec{G}(x,y) = (-x\vec{\imath} - y\vec{\jmath})/(x^2+y^2)$.

**Exercise 4.4.5** *Prepare a similar tables of values for the vector field* $\vec{L}(x,y) = (2x-3y),\vec{\imath} + (3x+y)\,\vec{\imath}$, $p=(2,1)$. *Again check at a few values how close to* linear *the table is. Use the table to read off numerical estimates for the partial derivatives. Explain how your findings relate to the statement that* the derivative of a linear function is the same linear function.

It is time to formally define differentiability and derivatives of a vector field. The intuitive, and fundamental, idea is that a vector field (just like any other function) is differentiable at a point, if after sufficient magnification it is practically indistinguishable from a linear field (function) (and remains upon further magnification). In first year calculus the slope of the graph of that linear function is called the derivative. More precisely, we shall call the linear function itself the derivative.
It is helpful to return briefly to first year calculus. Basically, instead of calling the number $m = f'(x_0)$ the derivative of $f$ at $x_0$, we now consider the linear function $f' = f'_{x_0}\colon dx \mapsto m\,dx$ the derivative. Other common, suggestive but less precise, notation for this function is $\Delta y = m\,\Delta x$. In the end, the key is to remember that any derivative $f'$ is a function that takes two inputs, the point $x_0$, and the increment $\Delta x$ or $dx$. First year calculus commonly focuses on how $f'$ changes at different points $x_0$. Sometimes to the extent that the main idea is almost lost ...: For fixed $x_0$, as a function of the increment $dx$, the derivative is a linear function.
Recall that not every function has a derivative. For example, no matter how much one zooms in on the graph of $f(x) = |x|$ at $x_0 = 0$, it never gets close to a straight line. Thus one says that $f(x) = |x|$ is not differentiable at $x_0 = 0$.

**Definition 4.4.1** A vector field $\vec{F}$ in the plane is called differentiable at a point $p = (x_0, y_0)$ if there exists a linear vector field $\vec{L}$ such that

$$\vec{F}(x_0+\Delta x, y_0+\Delta y) = \vec{F}(x_0,y_0) + \vec{L}(\Delta x, \Delta y) + o(\|(\Delta x, \Delta y)\|) \tag{4.24}$$

The divergence of the vector field $\vec{F}$ at the point $(x_0,y_0)$ is defined to be the *trace* of $\vec{L}$, i.e. the quantity $\vec{\imath}\cdot\left(\vec{L}(1,0) - \vec{\jmath}\cdot\vec{L}(0,1)\right)$ (i.e. $(a+d)$ in our usual notation).
The scalar curl of the vector field $\vec{F}$ at the point $(x_0,y_0)$ is defined to be twice the *skew symmetric part* of $\vec{L}$, i.e. the quantity $\left(\vec{L}(1,0)\cdot\vec{\jmath} - \vec{L}(0,1)\cdot\vec{\imath}\right)$ (i.e. $(c-b)$ in our usual notation).

**Definition 4.4.2** A vector field $\vec{F}$ in 3-space is called differentiable at a point $p = (x_0, y_0, z_0)$ if there exists a linear vector field $\vec{L}$ such that

$$\vec{F}(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) = \vec{F}(x_0,y_0,z_0) + \vec{L}(\Delta x, \Delta y, \Delta z) + o(\|(\Delta x, \Delta y, \Delta z)\|) \tag{4.25}$$

The divergence of the vector field $\vec{F}$ at the point $(x_0,y_0,z_0)$ is defined to be the *trace* of $\vec{L}$, i.e. the quantity $\left(\vec{\imath}\cdot\vec{L}(1,0,0) + \vec{\jmath}\cdot\vec{L}(0,1,0) + \vec{k}\cdot\vec{L}(0,0,1)\right)$.
The curl of the vector field $\vec{F}$ at the point $(x_0,y_0,z_0)$ is defined to be twice the *skew symmetric part* of $\vec{L}$, i.e. the vector

$$\left(\vec{k}\cdot\vec{L}(0,1,0) - \vec{\jmath}\cdot\vec{L}(0,0,1)\right)\vec{\imath} + \left(\vec{\imath}\cdot\vec{L}(1,0,0) - \vec{k}\cdot\vec{L}(0,0,1)\right)\vec{\jmath} + \left(\vec{\jmath}\cdot\vec{L}(1,0,0) - \vec{\imath}\cdot\vec{L}(0,1,0)\right)\vec{k}. \tag{4.26}$$

The numerical explorations suggest that the coefficients of the derivative(s) of the vector field $\vec{F}$ can be easily obtained as partial derivatives (evaluated at $(x_0, y_0)$).

**Proposition 4.4.1** *If a vector field $\vec{F}$ is differentiable at a point $p$ then the coefficients of its derivative agree with the components of the Jacobian matrix of partial derivatives of its component functions*

$$JF(x_0, y_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(x_0, y_0) & \frac{\partial F_1}{\partial y}(x_0, y_0) \\ \frac{\partial F_2}{\partial x}(x_0, y_0) & \frac{\partial F_2}{\partial y}(x_0, y_0) \end{pmatrix} \tag{4.27}$$

For most practical purposes one obtains $\vec{L} = \overrightarrow{DF}$ via this simple partial differentiation. Nonetheless, there exist vector fields $\vec{F}$, e.g. defined by $\vec{FF}(x, y) = xy\vec{\imath}/(x^2 + y^2)$ if $(x, y) \neq (0, 0)$ and $\vec{FF}(0, 0) = 0$ for which the partial derivatives are defined everywhere, but the vector field is not differentiable.

**Exercise 4.4.6** *Plot the vector field $\vec{F}$ of the preceding comment. Zoom in on it at the point $(0, 0)$ and verify that it does not converge to a linear field at this point. Then use the definition of partial derivatives, e.g. $\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{1}{h}(f(h, 0) - f(0, 0))$, to verify that the Jacobian matrix of partial derivatives $JF$ exists at $(0, 0)$, and is equal to the zero matrix.*

continuity
no proof or prop or of sufficiency
emphaisize linear approx as central idea
If the flux/trace case case was dealt with in previous section, include div in the same box/defn.
this might be the most effective way.)

**Exercise**: When zooming of the first kind at any point in a linear field what does one see? In particular, are linear fields differentiable? What is their scalar curl? How does the scalar curl change from point to point in a linear field. Is zooming required at all? 0mm

Several more exercises, mostly JAVAscope related. E.g. find a vector field that has positive scalar curl in the right half plane, and negative scalar curl in the left half plane.

Since the scalar curl $(c - b)$ of a linear field naturally appeared when analyzing line integrals, it is natural to ask how the scalar curl of differentiable nonlinear fields is related to line integrals.

**Class exercise**: Pick a *typical* nonlinear vector field, say $\vec{F}(x, y) = (\cos x + \sin y))\,\vec{\imath} + (\cos y\ \sin y)\,\vec{\jmath}$. Split the class into three groups. Each group picks a different *base point* in the plane, such as $(x_0, y_0) = (4, 7)$. Each student picks, or is assigned one or several different contours (just as in the class exercise in the section on linear fields, e.g. rectangles, squares, various triangles, circles, half circles, etc.) that is based at her/his base point.
Starting with a *diameter* (or side-length) $a = 1$ set up and evaluate the line integral of the field over the contour. Tabulate the results - keep many digits, don't truncate!

| Student name | Base point | Curve | Line integral | | | Ratio: $\frac{\text{Integral}}{\text{Area}}$ | | |
|---|---|---|---|---|---|---|---|---|
| | | | $a=1$ | $a=0.1$ | $a=0.001$ | $a=1$ | $a=0.1$ | $a=0.001$ |
| | (0,0) | Circle $C_1$ | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |

Just as the slopes of line segments connecting two points on the graph of a function of a single variable usually are different depending on the location of the points, and their distance from each other, here, too, we obtain ratios that change with the location and the size of the contours. However, as the contours become increasingly smaller the ratios approach a limit. The limit is the same for any shape at the same base point, but it varies from point to point. This is in complete analogy to single variable calculus where the difference quotients approach well-defined limits at each point, bit generally different limits at different points. Here, there is the added feature of also being able to consider the shapes. The explanation that the shapes do not matter is easy: On a sufficiently small scale, the differentiable field is practically linear, and we proved in the preceding section that for linear fields the ratio of the line integral divided by the area is independent of the shape of the curves considered!

It remains to obtain formulas for the scalar curl in the usual coordinate systems. In rectangular coordinates we simply consider the Taylor approximations of the $\vec{\imath}$ and $\vec{\jmath}$ components of the differentiable vector field at the point of interest: (*refer back to prior section or appendix*).

Need to sleep now.

$$\vec{F}(x_0+\Delta x, y_0+\Delta y) = \vec{F}(x_0,y_0) + \vec{L}(\Delta x,\Delta y) + o(\|(\Delta x,\Delta y)\|) \tag{4.28}$$

$$\vec{L}(\Delta x,\Delta y) = \left(\frac{\partial F_1}{\partial x}(x_0,y_0)\Delta x + \frac{\partial F_1}{\partial y}(x_0,y_0)\Delta y\right)\vec{\imath} + \left(\frac{\partial F_2}{\partial x}(x_0,y_0)\Delta x + \frac{\partial F_2}{\partial y}(x_0,y_0)\Delta y\right)\vec{\jmath} \tag{4.29}$$

For a formula for the scalar curl in polar coordinates it is helpful to consider a suitable contour in the plane

xxxx

the usual picture ???

## 4.5 Green's theorem for nonlinear fields

**Intuitively, precise statement**

Partition the region as before. But this time don't worry about shapes at all. On the other hand mae the regions all sufficiently small.

**approximate identities**

**error terms, and convergence criteria**

**usual induction proof, still need partitioning argument**

Main issue: Where to put the error estimates, in the preceding section – we already have $o(\|\Delta r\|)$ or into this section. maybe here put the epsilons for the Riemann sums if any at all
on small scale linear, work with approximate equal
make this an extremely short and elegant section. all the work was done in the linear case, and when zooming, maing the right definition. this should be a freebie

**Conjecture**: *For.*

## 4.6 Using Green's theorem

center of mass of flower shaped region convert into line inetgrals

Not for silly calculations

prove the curl test and analogue about path independence

homotop one curve into another

control examples

back to the magnetic field and induced voltage in time varying magnetic field? or moving loop.

## 4.7 Three dimensions

**Proposition 4.7.1** [Compare proposition 4.4.1 in two dimensions.] *If a vector field $\vec{F}$ is differentiable at a point p then the coefficients of its derivative agree with the components of the Jacobian matrix of partial derivatives of its component functions*

$$JF(x_0, y_0, z_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(x_0, y_0, z_0) & \frac{\partial F_1}{\partial y}(x_0, y_0, z_0) & \frac{\partial F_1}{\partial x}(x_0, y_0, z_0) \\ \frac{\partial F_2}{\partial x}(x_0, y_0, z_0) & \frac{\partial F_2}{\partial y}(x_0, y_0, z_0) & \frac{\partial F_2}{\partial 1}(x_0, y_0, z_0) \\ \frac{\partial F_3}{\partial x}(x_0, y_0, z_0) & \frac{\partial F_3}{\partial y}(x_0, y_0, z_0) & \frac{\partial F_3}{\partial 1}(x_0, y_0, z_0) \end{pmatrix} \tag{4.30}$$