Show your work - explain what you are doing. Scattered formulas without clear logical order and computer printouts without explanations will be ignored (ZERO CREDIT!) Check ALL calculations with MATLAB! It is YOUR responsibility to demonstrate that you have mastered the material of this class.

1. Find an invertible matrix $V$ that diagonalizes the matrix $A$ given on the right, i.e., find $V$ and $D$ such that $V^{-1}AV = D$ is diagonal. **Formulas and symbols alone will not earn much credit. Present a compelling “story”, laying out in detail all major steps, demonstrating that you understand why you are doing what.**

2. a. Show that similar matrices have the same eigenvalues. **Hint: Compare** $\det(\lambda I - S^{-1}AS)$ and $\det(\lambda I - A)$.
   
   b. Give an explicit counterexample that shows that generally $e^A e^B \neq e^{A+B}$.
   
   c. Calculate the matrix exponential of the matrix $A = \begin{pmatrix} 4/3 & 5/3 & -5/3 \\ 5/3 & 4/3 & 5/3 \\ -5/3 & 5/3 & 4/3 \end{pmatrix}$. Show all major steps. Check with $\text{expm}(A)$.

3. Suppose $A$ is a square $n \times n$ matrix. Give ten, as diverse as possible (every chapter!), statements, each of which is equivalent to “zero is not an eigenvalue of $A$”. You may want to revisit this after working problem 6.

4. In each of the following justify your reasoning – especially if it is impossible to determine $T(v)$.
   
   a. Suppose $T$ transforms $(2, 5)$ to $(-1, 1)$ and $(1, 3)$ to $(1, 1)$. Find $T(v)$ when $v = (0, 1)$.
   
   b. Suppose $T$ transforms $(2, 6)$ to $(-1, 1)$ and $(1, 3)$ to $(1, 1)$. Find $T(v)$ when $v = (-1, -3)$.
   
   c. Suppose $T$ transforms $(2, 6)$ to $(-1, 1)$ and $(1, 3)$ to $(1, 1)$. Find $T(v)$ when $v = (0, 1)$.

5. a. Find the point in the plane that contains $(0, 0, 0)$, $(1, 0, 3)$, and $(0, 1, 2)$ that is closest to the point $(3, 4, 5)$. 
   
   b. Find an orthonormal basis for the plane. **Briefly** explain what you are doing.
   
   c. Suppose that $u$ is an $n \times 1$ vector of unit length $u^T u = 1$. Show that $I_{n \times n} - 2uu^T$ is an orthogonal matrix.

6. Suppose $A$ is an $m \times n$ matrix with $m > n$ of maximal rank.
   
   a. Explain why $A^T A$ is invertible. **Hint: Show that if $A^T Ax = 0$ then $Ax = 0$.**
   
   b. Explain why there exist an orthogonal $V$ and a diagonal $\Sigma$ with positive real diagonal entries such that $A^T = V \Sigma^2 V^T$.
   
   c. Show that $U = AV \Sigma^{-1}$ has orthogonal columns.

7. Decide whether true or false. **Mark your answers T/F on the left margin of this sheet.** Provide a counterexample for at least 3 false statements, and **briefly** explain “why” for at least 2 true statements.
   
   a. If $A$ is similar to $B$ then $A^2$ is similar to $B^2$.
   
   b. Every square $n \times n$ matrix has $n$ linearly independent eigenvectors.
   
   c. If $A$ is a $3 \times 3$ matrix with positive trace and positive determinant then its eigenvalues are positive.
   
   d. If $A$ is a $2 \times 2$ matrix with positive trace and positive determinant then its eigenvalues are positive.
   
   e. If $A$ and $B$ are $m \times n$ and $n \times p$, then the columns of $AB$ are linear combinations of the columns of $B$.
   
   f. If $A$ and $B$ are $3 \times 1$ and $1 \times 3$ matrices, respectively, then det $AB = 0$.
   
   g. The diagonal elements of a triangular matrix are its eigenvalues.
   
   h. If $A$ and $B$ are symmetric (and of the same size), then $AB$ has real eigenvalues and an orthogonal basis of eigenvectors.
   
   i. If $A$ is an $n \times n$ matrix with zero determinant, then $Ax = 0$ has (at least) two different solutions.
   
   j. If $A$ is an $n \times n$ matrix with zero determinant, and $b$ an $n \times 1$ vector, then $Ax = b$ has no solution.