1. Consider the cube with vertices \((\pm 1, \pm 1, \pm 1)\), all possible combinations, labelled as in the diagram below. Let \(\beta = \{A, B, C\}, \gamma = \{D, E, H\}\), and \(\varepsilon = \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\}\).
   Let \(L: \mathbb{R}^3 \mapsto \mathbb{R}^3\) be the linear transformation defined by \(L(A) = H, L(B) = E\) and \(L(C) = F\).
   a. Is \(\beta\) a basis for \(\mathbb{R}^3\)? Justify your answer.
   b. Find the coordinates of \(F\) w.r.t. \(\beta\) (if possible).
   c. Find the matrices \(\beta[L]_\beta\) and \(\varepsilon[L]_\varepsilon\) representing \(L\) with respect to the basis \(\beta\), and with respect to the standard basis \(\varepsilon\).
   d. Calculate \(L(D)\).
   **Bonus:** Does there exist a linear trafo. \(T: \mathbb{R}^3 \mapsto \mathbb{R}^3\) such that (i) \(T(B) = A, T(D) = B\) and \(T(H) = C\), or such that (ii) \(T(B) = A, T(D) = E\) and \(T(H) = G\)? In either case, find a matrix representation for \(T\), or, if impossible, explain why.

2. Decide which of the following are linear. Demonstrate the linearity in detail in **two** cases, and provide an explicit counterexamples for **two** maps that are not linear.
   Find bases for the kernel and the range (image) of **two** linear maps each.
   a. \(M: \mathbb{R}^3 \mapsto \mathbb{R}\) defined by \(M: (x_1, x_2, x_3)^T \mapsto \max\{x_1, x_2, x_3\}\).
   b. \(L: \mathbb{R}^2 \mapsto \mathbb{R}^3\) defined by \(L: (x_1, x_2)^T \mapsto (x_1 + x_2, x_1 - x_2, x_1 - x_2)^T\).
   c. Let \(A \in \mathbb{R}^{3 \times 3}\) be a fixed nonsingular matrix. \(L: \mathbb{R}^3 \mapsto \mathbb{R}^3. L(b) = (\text{solution } x \text{ of } Ax = b)\).
   d. \(L: \mathbb{R}^{2 \times 2} \mapsto \mathbb{R}\) defined by \(L: A \mapsto \det(A)\).
   e. Let \(S \in \mathbb{R}^{3 \times 3}\) be a fixed nonsingular matrix. \(L: \mathbb{R}^{3 \times 3} \mapsto \mathbb{R}^{3 \times 3}\) defined by \(L: A \mapsto SAS^{-1}\).
   f. \(T_3: C^\infty(\mathbb{R}) \mapsto \mathbb{P}_3\) defined by \(T_3: f \mapsto f(0) + f'(0) \cdot x + \frac{1}{2} f''(0) \cdot x^2 + \frac{1}{6} f'''(0) \cdot x^3\).
   g. \(L: C^0(\mathbb{R}) \mapsto C^0(\mathbb{R})\) defined by \(L(f)(x) = \frac{1}{2}(f(x) + f(-x))\).

3. Consider the subset \(\beta = \{1, 1 + x, 1 + x + \frac{1}{2} x^2, 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3\} \subseteq \mathbb{P}_3\) of polynomial functions.
   a. Explain why \(\beta\) is a basis for \(\mathbb{P}_3\).
   b. Find the coordinates of \(p(x) = 1 + x + x^2 + x^3\) with respect to \(\beta\).
   c. Show that the map \(L: \mathbb{P}_3 \mapsto \mathbb{P}_3\) defined by \(L(p(x)) = p'(x)\) is linear.
   d. Find bases for the kernel and for the range (image) of \(L\).
   e. Find the matrix representation \(A = \beta[L]_\beta\) of \(L\) with respect to \(\beta\).
   f. Find the matrix representation \(B = \varepsilon[L]_\varepsilon\) of \(L\) w.r.t. the standard basis \(\varepsilon = \{1, x, x^2, x^3\}\).
   g. Find a matrix \(S\) such that \(A = SBS^{-1}\). If impossible, explain why.