Definitions:
A set $S$ is finite if it is empty or there exists $n \in \mathbb{N}$ and a bijection $f: S \mapsto [n]$. A set $S$ is countable if it is finite or there exists a bijection $f: S \mapsto \mathbb{N}$. A set $S$ is uncountable if it is not countable.

**Proposition.** Every subset $S \subseteq \mathbb{N}$ is countable.

**Corollary 1.** If $S$ is a set $f: S \mapsto \mathbb{N}$ is one-to-one then $S$ is countable.

**Corollary 2.** If $S$ is a set $f: \mathbb{N} \mapsto S$ is onto then $S$ is countable.

**Exercise.** Assuming the proposition, prove the corollaries.

**Proof** (of the proposition) (intuitively clear – but lots of technical details . . .)
Suppose $S \subseteq \mathbb{N}$ and $S$ is not finite. We will construct a bijection from $\mathbb{N}$ to $S$. It is convenient to prove a number of properties of the function being defined along with the actual recursive construction. More specifically, for each $n \in \mathbb{N}$ let $f_n: [n] \mapsto \mathbb{N}$ denote the restriction of $f$ to the set $[n]$. Along with defining each $f_n$ we shall prove that each

(i) $f_n$ is one-to-one, and
(ii) $f(n) \geq n$.

Since $S$ is nonempty (the empty set if finite!), there exists a smallest element in $S$ (using the WOP). Define $f(1)$ to be this element.

Suppose $f_n$ has been defined and satisfies (i) and (ii). Since $S$ is not finite, $f_n$ cannot be onto. Thus $S \setminus f([n]) = S \setminus \{f(1), \ldots f(n)\}$ is nonempty and by the WOP it has a smallest element. Define $f(n+1)$ to be this smallest element.

Since $f_n$ is one-to-one and $f(n+1) \neq f(k)$ for any $k \leq n$, it is clear that $f_{n+1}$ is one-to-one. Moreover, since $f(n) \geq n$ and the smallest element of $S \setminus f([n])$ must be larger than $f(n)$ (prove this!) it follows that $f(n+1) > (n+1)$.

To show that $f$ is one-to-one, suppose that $n, n' \in \mathbb{N}$ are such that $f(n) = f(n')$. W.l.o.g. assume that $n \geq n'$. Then $f(n') = f_{n'}(n')$ which together with $f_n(n) = f(n)$ yields that $f_n(n) = f_{n'}(n')$. But $f_n$ is one-to-one and hence $n = n'$.

[[This proves in generality the useful: Lemma. If $f: \mathbb{N} \mapsto Y$ is any function such that for every $n \in \mathbb{N}$ the restriction of $f$ to $[n]$ is one-to-one, then $f$ is one-to-one.]]

To show that $f$ is onto, suppose $n \in S$. Since $f(n+1) \geq (n+1) > n$ and $f(n+1)$ is the smallest element in $S \setminus f([n])$, it is clear that $n$ cannot be an element of $S \setminus f([n])$. Thus $n$ must be among the elements of $f([n]) = \{f(1), \ldots f(n)\}$, i.e. there exists $k \in [n]$ such that $f(k) = n$. ■