1.a The function \( f : \mathbb{R} \mapsto [1, \infty) \) defined by \( f(x) = x^2 + 1 \) is not one-to-one because \( f(-1) = 2 = f(1) \) but \(-1 \neq 1\). Conversely suppose \( b \) is an element in the codomain, i.e. \( b \geq 1 \). Let \( a = \sqrt{b-1} \). Since \( b \geq 1 \), the square root is well-defined and \( a \geq 0 \), i.e. is an element of the domain of \( f \). Since \( f(a) = f(\sqrt{b-1}) = (\sqrt{b-1})^2 - 1 = b \), this shows that \( f \) is onto.

**Comments:** First of all, the codomain matters. For example the function \( g : \mathbb{R} \mapsto \mathbb{R} \) defined by \( g(x) = f(x) \) is NOT onto. Secondly, showing that a function is onto is an “existence proof”. Please take a moment to review section 1.9. The first step is to produce a candidate – above this meant producing \( a = \sqrt{b-1} \). Carefully review the gray boxes on pages 28 and 29 – it may involve a LOT of scratch work (algebra!) to come up with this formula, but such work is NOT a necessary part of the proof (i.e. no matter how much work it takes, it does not earn MAT 300 credit!). Our concerns are to verify that it is a legal candidate (i.e., it is an element of the domain, here we need to verify that for every \( b \) in the codomain the square-root defines a real number). Then we have to show that the candidate does the job, here we have to verify that \( f(a) = b \).

This work could possibly correctly represented as a “two-column proof” (involving what we called scratch-work) – but it will require lots of words or very carefully placed arrows and quantifiers. So far I have not seen ANY two-column proof in our class that was acceptable as a proof. My best advice is to stay away from two-column arguments like the plague (unless you are really good at writing them correctly, e.g. if problem 5.2.6. is completely obvious to you.)

1.b The function \( f : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N} \) defined by \( f(n) = (n, n) \) is not onto because \( f(n) \neq (0, 1) \) for all \( n \in \mathbb{N} \). However, the function \( f \) is one-to-one because if \( n, m \in \mathbb{N} \) are such that \( f(n) = f(m) \), i.e. \( (n, n) = (m, m) \) then clearly \( n = m \).

2.b We show that for every \( k > 0 \) the function \( f : (-k^2, \infty) \mapsto (-\infty, k) \) defined by \( f(x) = kx/(x + k^2) \) is both one-to-one and onto. Suppose \( x_1, x_2 \geq -k^2 \) and \( f(x_1) = f(x_2) \). This implies that \( kx_1(x_2 + k^2) = kx_2(x_1 + k^2) \) and hence \( x_1x_2 = x_2k^3 \). Since \( k > 0 \) this implies that \( x_1 = x_2 \), showing one-to-leness of \( f \). Now suppose \( y < k \). Let \( x = -yk^2/(y - k) \). Since \( y \neq k \) this defines a real number. We need to argue that \( x \) indeed lies in the domain of \( f \): Since \( k > 0 \), clearly \( y > y - k \). Also, \( y < k \) means that \( y - k < 0 \), and thus \( 1/y < 1 \). Therefore, \( \frac{-yk^2}{y-k}(-k^2) > (-k^2) \), i.e. \( y \in \text{dom} f \). It is straightforward to verify that

\[
f(x) = f\left(\frac{-yk^2}{y-k}\right) = \frac{k \cdot \left(\frac{-yk^2}{y-k}\right)}{k^2} = -\frac{yk^3}{yk^2 + k^2(y-k)} = y = \frac{-yk^3}{-k^3} = y
\]

**Comment:** Remember the gray boxes on pages 28 and 29 – doing correct algebra (like coming up with the formula for \( x = f^{-1}(y) \)) is what you got algebra credit for, long ago. MAT 300 is about getting the arguments right.

3.a To get any idea what this is about consider some examples first. E.g. If \( A = B = \mathbb{R} \) consider \( S \) consisting of a point, a curve, the region inside or above a curve etc. Sketch the sets \( S \) and e.g. \( \{\Pi_1(x,y); (x,y) \in S\} \). Then also consider the case of finite sets, e.g. \( A = \{\heartsuit, \spadesuit\} \) and \( B = \{1,2,3\} \), and e.g. \( S = \{(\heartsuit,2), (\spadesuit,2), (\heartsuit,3)\} \). **Circle these elements NOW (on the right).** Calculate e.g. \( \Pi_1(a,b) \) for each \( (a,b) \in S \). Now you should have some idea about what might be true and what not – time to try to prove your conjectures.

3.b If \( S \subseteq A \times B \) is a function from \( A \) to \( B \) (alas, the graph of a function), then \( \Pi_1 \) is necessarily both one-to-one and onto, while \( \Pi_2 \) may be either, both, or neither. If \( S \) is a function from \( A \) to \( B \) then for every \( a \in A \) there exists exactly one \( b \in B \), namely \( b = S(a) \) such that \( (a,b) \in S \). Consequently \( \Pi_1(a,f(a)) = a \), showing that \( \Pi_1 \) is a one-to-one correspondence. Conversely, the following functions \( S_k \subseteq \mathbb{R} \times \mathbb{R} \), also written as \( S_k : \mathbb{R} \mapsto \mathbb{R} \), \( k = 1,2,3,4 \), show that anything may happen with \( \Pi_2 \):

- If \( S_1 : x \mapsto x \) then \( \Pi_2 \) is a one-to-one correspondence between \( S_1 \) and \( B \).
- If \( S_2 : x \mapsto e^x \) then \( \Pi_2 \) is one-to-one from \( S_2 \) to \( B \) (because \( e^x \cdot x = e^x \cdot 1 \) only if \( x = x_2 \)), but not onto \( B \) (e.g. \( \forall x, e^x \neq 0 \)).
- If \( S_3 : x \mapsto x^3 - x \) then \( \Pi_2 \) is onto \( B \) but not one-to-one since e.g. \( (0,0) \in S_3 \) and \( (1,0) \in S_3 \) and \( \Pi_2(0,0) = 0 = \Pi_2(1,0) \).
- If \( S_4 : x \mapsto 0 \) then \( \Pi_2 \) is neither one-to-one nor onto \( B \) since e.g. \( 1 \neq \Pi_2(a,b) \) for any \( (a,b) \in S_4 \), while \( \Pi_2(0,0) = \Pi_2(1,0) = 0 \) yet \((0,0)\) and \((1,0)\) are two different elements of \( S_4 \).