**MAT 300  Mathematical Structures**

**Equivalence classes and partitions**

**Definition:** Suppose $X$ and $Y$ are sets. A *relation from $X$ to $Y$* is a subset of $X \times Y$. A *relation on $X$* is a subset of $X \times X$.

If $R \subseteq X \times X$ is a relation and $(x, y) \in R$ one customarily writes $xRy$ or $R(x, y)$.

**Definition:** A relation $R$ on a set $X$ is an *equivalence relation* if it is

(i) reflexive: $(\forall x \in X) (x, x) \in R$,
(ii) symmetric: $(\forall x, y \in X) ((x, y) \in R) \rightarrow ((y, x) \in R)$, and
(iii) transitive: $(\forall x, y, z \in X) ((x, y) \in R) \land ((y, z) \in R) \rightarrow ((x, z) \in R)$.

If $R \subseteq X \times X$ is an equivalence relation it is customary to write $x \sim y$ for $(x, y) \in R$.

**Definition:** Suppose $\sim$ is an equivalence relation on a set $X$. The set $T_x = \{ y \in X : x \sim y \}$ is called the “equivalence class of $x$” (under $\sim$). It is customarily denoted by $[x]$ (or $[x]_\sim$).

**Definition:** A *partition* of a set $X$ is a collection $\mathcal{C} \subseteq \mathcal{P}(X)$ of subsets of $X$ such that every set $C \in \mathcal{C}$ is nonempty, and every $x \in X$ is a member of exactly one set $C \in \mathcal{C}$.

The $2^{\text{nd}}$ condition is often written as $(\forall C, D \in \mathcal{C}) ((C \cap D \neq \emptyset) \Rightarrow (C = D))$ together with $\bigcup_{C \in \mathcal{C}} C = X$.

**Theorem 1:** Suppose $\sim$ is an equivalence relation on a set $X$. Then the set $\mathcal{C} = \{ [x] : x \in X \}$ of equivalence classes of $\sim$ forms a partition of $X$.

**Theorem 2:** Suppose $\mathcal{C} \subseteq \mathcal{P}(X)$ is a partition of a set $X$. Then the relation $\sim$ on $X$ defined by $x \sim y \iff (\exists C \in \mathcal{C})(x, y \in C)$ is an equivalence relation.

**Proof** (of theorem 1). Suppose $\sim$ is an equivalence relation on a set $X$ and $\mathcal{C} = \{ [x] : x \in X \}$ is the associated set of equivalence classes.

- Every $C \in \mathcal{C}$ is the equivalence class of some $x \in X$. Then $x \in [x] = C$ implies $C \neq \emptyset$.
- Now suppose $C, D \in \mathcal{C}$. Then there are $x, z \in X$ such that $C = [x]$ and $D = [z]$. Suppose that $C \cap D \neq \emptyset$, i.e. there exists $y \in C \cap D$, i.e. $y \sim x$ and $y \sim z$. If $u \in C$ then $u \sim x$, $x \sim y$, $y \sim z$ and by transitivity of $\sim$ it follows that $u \sim z$, i.e. $u \in D$ and $C \subseteq D$. Analogously $D \subseteq C$ and hence $C = D$.
- Finally, if $x \in X$ then obviously $x \in [x] \in \mathcal{C}$ and thus $x \in \bigcup_{C \in \mathcal{C}} C$.

**Proof** (of theorem 2). Suppose $\mathcal{C} \subseteq \mathcal{P}(X)$ is a partition of $X$ and $\sim$ is the relation on $X$ defined by $x \sim y$ if there is a set $C \in \mathcal{C}$ such that $x, y \in C$.

- Suppose $x \in X$. There exists a set $C \in \mathcal{C}$ such that $x \in C$ and hence $x \sim x$.
- Suppose $x, y \in X$ and $x \sim y$. This means that there exists $C \in \mathcal{C}$ such that $x, y \in C$. Clearly this implies that also $y \sim x$.
- Suppose $x, y, z \in X$ and $x \sim y$ and $y \sim z$. This means that there exist $C, D \in \mathcal{C}$ such that $x, y \in C$ and $y, z \in D$. Since $y \in C \cap D$ it follows that $C = D$. Thus $x$ and $z$ are elements of the same set in $\mathcal{C}$ and hence $x \sim z$.