Instructions: There are more problems on the back side!
Do not waste time with the computers - maybe you do not want to use them at all.

1.) Match each of the following functions given by a formula to the corresponding table of numerical values, graph and/or contour maps. (Note: Some formulas may match none or more than one of the choices.)
In each case give at least one reason (be brief) why you think your choice is correct.

A. \( f(x, y) = x^2 - y^2 \)  
B. \( f(x, y) = 6 - 2x + 3y \)  
C. \( f(x, y) = \sqrt{1 - x^2 - y^2} \)  
D. \( f(x, y) = \frac{1}{1+x^2+y^2} \)  
E. \( f(x, y) = 6 - 2x - 3y \)  
F. \( f(x, y) = \sqrt{x^2 + y^2} \)

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2.a.) For the surface number 9, given above roughly (!) sketch the cross-sections parallel to the x-axis with \( y = 0 \), \( y = 1 \), and \( y = 2 \).

b.) What is the slope of the contour lines of the function \( f(x, y) = 6 - 2x + 3y \) in B.? Why?

3.a.) Describe in words the shape of the graph of \( f(x, y) = x^2 - 2y \).

b.) Sketch a contour map for \( f(x, y) \).

c.) Find a formula for the contour that passes through the point \((x, y) = (3, 1)\).
4.) Consider the chart below giving (in thousands) the number \( f(x, y) \) of grape vines of age \( x \) in year \( y \).

a.) Describe in words the most striking pattern in the table.
b.) Explain why it is reasonable that usually the number in any box equals the number diagonally above (to the left). Write an equation to formally express this relation.
c.) In one year a fungal disease killed most of the older grapevines, and in the following year a long freeze killed most of the young vines. Which are these years?
d.) Why are there no ten year old vines in 1992?
e.) Fix the year \( y = 1986 \) and sketch the graph for the age distribution \( g(x) = f(x, 1986) \) of grapevines in that year. Repeat for \( y = 1987 \). Describe and explain your observation.
f.) In 1986 a successful advertising campaign led to a dramatic increase in demand for premium wines. The growers followed by adding many more plants. Suppose a vine (the plant) produces the first harvestable grapes at age five, and is removed after sixteen years. How many (thousand) grape vines that bear fruit were there in the year 1986 and how many will be there in the year 1994?

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5a.) Consider the contour map given above which represents altitude in a mountain range. How high is the highest point on the map? Where is it? Which side of the mountain top is the steepest?
b.) Suppose that there is a spring at the point (5, 5). Carefully draw the path that the creek coming from this spring will take. At what angle does the creek cross the contours? Why?
c.) Sketch a cross-section (of the elevations) of the mountain range with \( y = 2 \) held constant.

For bonus credit: Find the maximum value of the function \( f(x, y) = \int_0^y \sin \frac{t}{y} \, dt \). (Hint: Look at the graph of \( g(t) = \frac{\sin t}{t} \)).
1.) Find as all matches between the following functions given by formulas, tables of numerical values, 3D-graphs, contour maps, or by names.

A. \( f(x, y) = 10 - 2x - 5y \)
B. \( f(x, y) = x^2 + y^2 \)
C. \( f(x, y) = x \)
D. \( f(x, y) = 10 + 2x + 5y \)
E. \( f(x, y) = x^2 - y^2 \)
F. \( f(x, y) = y \)
G. \( f(x, y) = 10 - 5x - 2y \)
H. \( f(x, y) = \sqrt{x^2 + y^2} \)
J. \( f(x, y) = (y - x)^2 \)
K. \( f(x, y) = 10 + 5x + 2y \)
L. \( f(x, y) = \sqrt{8 - (x^2 + y^2)} \)
M. \( f(x, y) = xy \)

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2.) Consider the function \( f(x, y) = x^2 y \).
   a.) Describe the shape of the graph in words.
   b.) Sketch, and describe in words cross-sections of the graph parallel to the x-axis, e.g. for \( y = -1, 0, 1, 2 \).
   c.) Sketch, and describe in words cross-sections of the graph parallel to the y-axis, e.g. for \( x = -1, 0, 1, 2 \).
   d.) Sketch, and describe in words the contours, e.g. for \( z = -1, 0, 1, 2 \).
   e.) Find a formula for the contour that passes through the point \((x, y) = (2, 3)\), and calculate the slope of this contour at this point.

3.) Consider the cube with side-length one, placed in a coordinate system as shown.
   a.) Express the vectors \( DA \) and \( DF \) in terms of the basis vectors \( \vec{i}, \vec{j}, \) and \( \vec{k} \).
   b.) Find a vector perpendicular to the triangle \( ADF \), find the area of the triangle, and write down the equation for the plane that contains the triangle.
   c.) Does the point \( P(3, 23, 26) \) lie in the same plane? Explain!
   d.) Find the angle \( AFD \) (i.e. the angle in the triangle at the corner \( F \)).
4. a) Consider the contour map of a function \( f(x,y) \) given below which represents altitude in a mountain range.
   a.) Find the highest and the lowest point. How high are they?
   b.) Find the steepest point on the map. Sketch a vector at this point pointing in the steepest direction up. Estimate its \( \mathbf{i} \) and \( \mathbf{j} \) components.
   c.) If there is a spring at the point \( S \), mark the path water running from this spring will take. At what angle will the path cross the contours? Why?
   d.) Sketch a cross-section (of the elevations) of the mountain range with \( y = 1.6 \) held constant (i.e. from \( A \) to \( B \)).

5.) For a triangle with sides of lengths \( a, b, \) and \( c \) the law of cosines asserts that \( c^2 = a^2 + b^2 - 2ab \cos \gamma \) where \( \gamma \) is the angle at the point \( C \), opposite to the side with length \( c \).
   a.) Sketch a triangle and label it appropriately.
   b.) Explain how one can easily get the law of cosines from the dot product.
   
   Hint: Introduce vectors \( \vec{v} = \overrightarrow{CA} \) and \( \vec{w} = \overrightarrow{CB} \), and consider the dot-product \((\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v})\).
2. Both $x$ and $y$ axes are part of the graph. Cross-sections parallel to the $xz$-plane are parabolas that get tighter for larger absolute $y$-values. The vertices of the parabolas are at zero, and the parabolas open upward for $y$ positive and downward for $y$ negative.

\[ z = c \iff y = \frac{c}{x^2} \]

\[ f(2, 3) = 12 = c \]

\[ y = \frac{12}{x^2}, \quad \frac{dy}{dx} = -\frac{24}{x^3} \]

at $(2, 3)$ \[ \frac{dy}{dx} = -3 \]

3. a) $\vec{DA} = \vec{i}$, $\vec{DF} = \vec{i} + \vec{j} + \vec{k}$. b) $\vec{N} = \vec{DA} \times \vec{DF} = \vec{k} - \vec{j}$ is perp. to triangle.

area $= \frac{1}{2} |\vec{N}| = \frac{1}{2} |\vec{r}|$. plane: $z - y = 0$ or $z = y$.

c) $\vec{p}$ does not lie on the plane since $23 \neq 26$.

d) $\cos \alpha = \frac{\vec{FA} \cdot \vec{FD}}{|\vec{FA}| |\vec{FD}|} = \frac{(-\vec{j} - \vec{k}) \cdot (-\vec{i} - \vec{j} - \vec{k})}{\sqrt{2} \cdot \sqrt{3}} = \frac{-1}{\sqrt{6}} \approx 0.6155$ i.e. $\alpha \approx 35.26^\circ$

4. a) Highest: near $(1.8, 4.7)$, over 1000 high.

Lowest: near $(1, 1.8)$, below 100 high.

b) Steepest: near $(3.9, 1.4)$, steepest direction $\vec{u} + 5\vec{v}$

c) see picture, flow towards right top.

d) 1000

5. $\vec{w} \cdot \vec{v} = \vec{w} \cdot \vec{v} = (\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v})$

\[ = \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{v} = |\vec{w}|^2 - 2|\vec{v}| |\vec{w}| \cos \theta + |\vec{v}|^2 \]

\[ = b^2 + a^2 - 2ab \cos \theta \]

Using only the definition of the dot product and its distributivity.
1.) Consider the function \( f(x, y) = \sqrt{x^2 + y^2} \).
   a.) Describe the shape of the graph in words.
   b.) Sketch, and describe in words, cross-sections of the graph parallel to the \( x \)-axis, e.g. for \( y = -1, 0, 1, 2 \).
   c.) Sketch, and describe in words the contours, e.g. for \( z = 0, 1, 2 \).
   d.) Find a formula for the contour that passes through \((x, y) = (3, 4)\); calculate the slope of this contour at this point.

2.) Find all matches between the following functions given by formulas, tables of numerical values, 3D-graphs, contour maps, or names of their graphs. (I found six three-way matches, plus two pairs).

   A. \( f(x, y) = x \)  
   B. \( f(x, y) = x^2 + y^2 \)  
   C. \( f(x, y) = (y + x)^2 \)  
   D. \( f(x, y) = -x \)  
   E. \( f(x, y) = \sqrt{x^2 + y^2} \)  
   F. \( f(x, y) = (y - x)^2 \)  
   G. \( f(x, y) = x + y \)  
   H. \( f(x, y) = y^2 - x^2 \)  
   J. \( f(x, y) = \frac{1}{x+y} \)  
   K. \( f(x, y) = y - x \)  
   L. \( f(x, y) = xy \)  
   M. \( f(x, y) = \frac{1}{x^2+y^2} \)

1. A plane  
2. A (half) sphere  
3. A circular cone  
4. A circular paraboloid  
5. A saddle

3.) Consider the contour map of a function \( f(x, y) \) given on the left, representing altitude in a mountain range (in hundreds of meters).
   a.) Find the highest point. How high is it?
   b.) Find the steepest point on the map (there are several reasonable choices). Sketch a arrow (vector) at this point pointing in the steepest direction down.
   c.) If there is a spring at the point \( S(3, 2) \), mark the path water running from this spring will take. At what angle will the path cross the contours? Why?
   d.) Sketch a cross-section (of the elevations) when \( y = 3 \) is held constant (i.e. from \( A \) to \( B \)).
1.a.) The graph is a circular cone, opening along the positive $x$-axis at an angle of $\frac{\pi}{4}$.

b.) The cross-sections are hyperbolas for $y \neq 0$ and the V-shaped graph of the absolute value function for $y = 0$.

c.) The contours have equations $x^2 + y^2 = 0, 1, 4$, are the point $(0,0)$, and circles centered at $(0,0)$ with radii $r = 1, 2$.

d.) $f(3,4) = 5$ and the contour is $x^2 + y^2 = 25$. Its slope at $(3,4)$ is $y' = \frac{dy}{dx} = -\frac{3}{4}$ (e.g. implicit differentiation).

2.) $(6,1,K), (7,5,H), (8,3,E), (9,1,D), (10,4,B), (11,14), (12,F), (13,5,L)$. (Plus some explanations.)

3.a.) The highest point is near $(3,0)$, its height is above 1700m. b.) The steepest points are just below the highest point, somewhat to the right; the steepest direction down is perpendicular to the contours, away from the high point.

c.) Crossing the contours at right angles the water will run towards the point $(-2,5)$. It crosses at right angles because this is the steepest way down. d.) See graph above.

The contours represent the following function (using MAPLE notation): $g := (x,y) \rightarrow f(x,y) + 1.6*f(x-3,y) + 1.1*f(x-4,y-3) + f(x-1,y-5)/1.3 + 0.8*f(x+3,y-3) + f(x+4,y+3)/1.3 + f(x+2,y+2)/2 - f(x-1,y-4)/3$.

Here $f := (x,y) \rightarrow 1/(1+x^2+y^2)^2$, whose graph is a single bump at $(0,0)$. $g$ is a sum of many bumps (volcanos), obtained by adding translates of $f$ and scaling their heights. A 3D-view is given below, and also the plot of the gradient field of $-g$ is shown. This shows the direction the water flows from any point. Gradients are a key topic in this course, will be introduced in Chapter 13, and used throughout the rest of the semester.
1.) Consider the parametrized curve defined by \( x = e^t \cos t \) and \( y = e^t \sin t \).
   a.) Describe the shape and interesting features of the curve. Sketch the curve.
   b.) Calculate the velocity \( \vec{v}(t) \) and the acceleration \( \vec{a}(t) \).
   c.) Find the position \( \vec{r}(0) \), velocity \( \vec{v}(0) \), acceleration \( \vec{a}(0) \) at time \( t_0 = 0 \). Sketch these vectors.
   d.) Find the next time \( t_1 \) at which the curve again crosses the positive \( x \)-axis, and calculate the position \( \vec{r}(t_1) \), velocity \( \vec{v}(t_1) \), acceleration \( \vec{a}(t_1) \) at this time. Sketch these vectors.
   e.) Calculate the length of the curve between the two times \( t_0 = 0 \) and \( t_1 \).
   f.) Calculate and sketch the graphs of the speed \( |\vec{v}(t)| \), the parallel and perpendicular components of the acceleration \( \vec{a}_\parallel(t) \), and \( \vec{a}_\perp(t) \), and the curvature \( \kappa(t) \).
   g.) Is the angle between the acceleration and the velocity vector smaller or larger than \( \frac{\pi}{2} \)? Explain what this tells you about the speed. Is this angle increasing or decreasing?

2.) Choose either one of the following:
   a.) Consider the following lists of equations

\[
0 = \frac{d}{dt} 1 = \frac{d}{dt} \left( \vec{T}(t) \cdot \vec{T}(t) \right) = \left( \frac{d}{dt} \vec{T}(t) \right) \cdot \vec{T}(t) + \vec{T}(t) \cdot \left( \frac{d}{dt} \vec{T}(t) \right) = 2 \left( \frac{d}{dt} \vec{T}(t) \right) \cdot \vec{T}(t)
\]

i.) Explain what this proves when suitably interpreted.
ii.) In detail explain each step, and justify why it can be performed.

or:

b.) Let \( A, B, C \) be a triangle with longest side \( AB \), and let \( M \) be the midpoint of this side \( AB \). Use vectors, and (properties of) dot-products to verify the theorem of Thales: The triangle is a right triangle if and only if the point \( C \) lies on the circle with center \( M \) passing through \( A \) and \( B \). (Suggestion: Let \( \vec{v} = \vec{AM} = \vec{MB} \), \( \vec{w} = \vec{MC} \) and consider \( \vec{v} \cdot (\vec{v} + \vec{w}) \cdot (-\vec{v} + \vec{w}) \).)

3.) Let \( T \) be the triangle with corners at \( A(2, 7, 1) \), \( B(-3, -1, 3) \) and \( C(4, -4, 4) \).
   a.) Find a vector perpendicular to the triangle.
       Calculate the area of \( T \).
       Find a numerical value for the interior angle \( \alpha \) at the point \( A \).
   b.) Write down the equation of the plane \( P \) containing the triangle \( T \).
   c.) Does the point \( D(1, 1, 1) \) lie in this plane? Justify your answer.
       If not, find the distance of the point \( D \) from the plane \( P \).
Show your work - explain what you are doing.
It is YOUR responsibility to demonstrate that you have mastered the material of this class.

1. In the following a, b, c, t, and f(t) denote scalars, \( \mathbf{r}(t), \mathbf{u}, \mathbf{v}, \mathbf{w} \), are vectors in 3-dimensional space, and \( \mathbf{A} \) is a 2 x 3 matrix, \( \mathbf{B} \) a 3 x 3 matrix and \( \mathbf{C} \) a 3 x 2 matrix.

For each of the following expressions decide whether it is a scalar, vector, matrix or an illegal expression. If illegal, point out why. If legal, rewrite the expression in at least one different way ("simplify", or the opposite); if it is a matrix, list its size, also.

\[
\begin{align*}
|\mathbf{u}| + \mathbf{v} \cdot \mathbf{w} & \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} & \quad \mathbf{t} + \mathbf{v} \cdot \mathbf{u} & \quad |\mathbf{r}'(t)| \times (\mathbf{bB}) & \quad (\mathbf{AB})\mathbf{v} & \quad \mathbf{AC} - \mathbf{CA} \\
(\mathbf{u} \times \mathbf{v}) \times \mathbf{v} & \quad (\mathbf{u} \cdot \mathbf{v}) \times \mathbf{v} & \quad \mathbf{t} + \mathbf{v} \times \mathbf{u} & \quad \mathbf{r}'(t) \times \mathbf{r}''(t) & \quad (\mathbf{CB})\mathbf{A} & \quad \mathbf{CA} + \mathbf{BB}
\end{align*}
\]

2. Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{12} \) denote the vectors from the center of a clock to the hours.
Express the vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) as linear combinations of \( \mathbf{v}_3 \) and \( \mathbf{v}_{12} \).
Find the angle between the lines from 8 o'clock to 1 o'clock, and to 2 o'clock, respectively.

3. For the three points \( \mathbf{P}(1,1,3), \mathbf{Q}(2,1,3), \) and \( \mathbf{R}(1,2,3) \) find
a. a parametric equation of the line through the points \( \mathbf{P} \) and \( \mathbf{Q} \)
b. a vector perpendicular to both the lines through \( \mathbf{P} \) and \( \mathbf{Q} \), and through \( \mathbf{P} \) and \( \mathbf{R} \)
c. an equation of the plane which contains the three points \( \mathbf{P}, \mathbf{Q}, \) and \( \mathbf{R} \)
d. the area of the triangle with corners \( \mathbf{P}, \mathbf{Q}, \) and \( \mathbf{R} \).

4.a. Describe the parameterized curve \( \mathbf{r}(t) = (3 \cos(t), 4 \sin(t), t) \) in words.
b. Calculate the velocity vectors \( \mathbf{r}'(t) \) and acceleration vectors \( \mathbf{r}''(t) \) as functions of time.
c. Sketch the graph of the speed \( |\mathbf{r}'(t)| \) as a function of time.
d. Calculate the angles between the velocity and acceleration vectors at time \( t = \pi / 2 \).
e. For bonus credit: At which points does the acceleration vector "point forward"? show your calculations, use a picture for illustration, and explain.

5. Let \( \mathbf{A} \) be the matrix \{ EMBED Equation.2 \} and consider the vectors \{ EMBED Equation.2 \}.
Calculate the "images" \( \mathbf{A}_{i}, \mathbf{A}_{j}, \mathbf{A}_{v}, \) and \( \mathbf{A}_{w} \).
Sketch (and label) all eight vectors.
Describe (make a reasonable guess) what \( \mathbf{A} \) does to a vector.
Calculate the determinant of \( \mathbf{A} \).
1. **Illegal algebraic operations.**
- \([\vec{u} + \vec{v} \cdot \vec{w} = \vec{u} + \vec{w} + \vec{v}]\) is a scalar (dot-product is commutative)
- \((\vec{u} \times \vec{v}) \times \vec{w} = -\vec{w} \times (\vec{u} \times \vec{v})\) is a vector (cross-product is anti-commutative)
- \((\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{u}) \cdot \vec{w} = 0\) is a scalar (triple product unchanged under cyclic permutations)
- \((\vec{u} \cdot \vec{v}) \times \vec{w} = (\vec{u} \times \vec{v}) \cdot \vec{w}\) is illegal (unless we understood the cross as scalar-multiplication, vector then).
- \(\vec{u} + \vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{w} + \vec{u}\) is a scalar (understanding that, as usual, multiplications are executed before additions).
- \(\vec{u} \cdot \vec{v} \times \vec{w}\) is illegal: We do not add vectors and scalars.
- \(|\vec{r}(t)| \times |\vec{y}(t)|\) is illegal (unless we understood the cross as scalar-multiplication of a matrix, \(3 \times 3\)-matrix then).
- \(|\vec{r}(t) \times \vec{r}(t)| = -|\vec{r}(t)| \times |\vec{r}(t)|\) is a vector.
- \((\vec{A}\vec{B})\vec{v} = \vec{A}((\vec{B}v))\) is a two-dimensional column vector (if we agreed to write \(\vec{v}\) as a three-dimensional column vector), using associativity of matrix multiplication (associativity of composition of maps).
- \((\vec{B}\vec{A})\vec{A}\) is illegal (the inner dimensions of \(\vec{C}\) and \(\vec{B}\) do not agree).
- \((\vec{A}^{-1} - \vec{C})\vec{A}\) is illegal as the two (legal) products result in matrices of different dimensions (\(2 \times 2\) and \(3 \times 3\)).
- \((\vec{C} + \vec{B})\vec{B} = \vec{B}^2 + \vec{C}\vec{A}\vec{A}\) is a \(3 \times 3\) matrix (using commutativity of matrix addition).

2. **The clock.**

The vectors \(\vec{v}_3\) and \(\vec{v}_{12}\) play the role of the usual basis vectors \(\hat{i}\) and \(\hat{j}\). Note that the absolute lengths of any of the vectors do not play any role at all. We agree that the angle between any subsequent vectors \(\vec{v}_1\) and \(\vec{v}_{12}\) is \(\pi/6\), and that all vectors \(\vec{v}_2\) have the same length.

Using elementary trigonometry rewrite the vectors \(\vec{v}_1\) and \(\vec{v}_2\) as linear combinations of \(\vec{v}_3\) and \(\vec{v}_{12}\):

\[
\vec{v}_1 = \cos\left(\frac{\pi}{6}\right)\vec{v}_3 + \sin\left(\frac{\pi}{6}\right)\vec{v}_{12} = \frac{1}{2}\vec{v}_3 + \frac{\sqrt{3}}{2}\vec{v}_{12} \quad \text{and} \quad \vec{v}_2 = \cos\left(\frac{\pi}{3}\right)\vec{v}_3 + \sin\left(\frac{\pi}{3}\right)\vec{v}_{12} = \frac{\sqrt{3}}{2}\vec{v}_3 + \frac{1}{2}\vec{v}_{12}
\]

Clearly \(\vec{v}_3 = -\vec{v}_2\). Thus we have to compute the angle between

\[
\vec{u} = \vec{v}_1 - \vec{v}_2 = \vec{v}_1 + \vec{v}_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} (\vec{v}_3 + \vec{v}_{12}) \quad \text{and} \quad \vec{w} = \vec{v}_2 - \vec{v}_3 = 2\vec{v}_2 - \vec{v}_3 = \vec{v}_3 + \sqrt{3}\vec{v}_{12}
\]

Utilize that for any two vectors \(\vec{u}\) and \(\vec{w}\) their dot-product is defined as \(\vec{u} \cdot \vec{w} = |\vec{u}| \cdot |\vec{w}| \cdot \cos \alpha\) where \(\alpha\) is the (desired) angle between the vectors. Thus, using the orthogonality of \(\vec{v}_3\) and \(\vec{v}_{12}\):

\[
\cos \alpha = \frac{\frac{1}{2}(1 + \sqrt{3})^2}{\frac{1}{2}(1 + \sqrt{3}) \sqrt{2} \cdot 2} = \frac{2 + \sqrt{3}}{2(1 + \sqrt{3})} \quad \text{and thus} \quad \alpha = \frac{\pi}{12}
\]

This result is also easily obtained using simple trig - draw a triangle with corners at 0,1 and 8. The angle at 0 must be 150° (by looking at the triangle with corners 0,1 and 2). The other two angles are equal, and thus must be 15°.

3. **Planes etc.**

a. Using that \(\vec{PQ} = \vec{i}\) a parametric equation for the line through \(P\) and \(Q\) is \(\vec{r}(t) = (1 + t, 1, 3)\).

b. Using that \(\vec{PR} = \vec{j}\), a vector perpendicular to the plane is \(\vec{N} = \vec{PQ} \times \vec{PR} = \vec{k}\).

c. Let \(\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}\) denote the vector to a generic point on the plane.

Then \(\vec{N} \cdot \vec{R} = \vec{N} \cdot \vec{OP}\), i.e. \(z = 3\), is an equation for the plane.

d. The area of the triangle is \(A = \frac{1}{2} \left| \vec{PQ} \times \vec{PR} \right| = \frac{1}{2}\).

4. **Curves.**

a. The curve is a helix (with constant pitch) winding around the z-axis. The top view shows an ellipse with semi-axes 3 and 4.

b. \(\vec{r}'(t) = (-3 \sin t, 4 \cos t, 1)\) and \(\vec{r}''(t) = (-3 \cos t, -4 \sin t, 0)\).

c. The speed \(|\vec{r}'(t)| = \sqrt{9 \sin^2 t + 16 \cos^2 t + 1} = \sqrt{10 + 6 \cos^2 t}\) oscillates between \(\sqrt{10} \approx 3.16\) and 4 (but it is not a sinusoidal). Use a calculator/computer to plot it.

4d. At \(t = \frac{\pi}{2}\) the dot-product between the velocity \(\vec{r}'(\pi/2) = (-3, 0, 1)\) and the acceleration \(\vec{r}''(\pi/2) = (0, 4, 0)\) clearly is zero, and thus the angle between the two vectors is \(\frac{\pi}{2}\).

4. **Matrices.**

a. A straightforward calculation gives \(A_i = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\), \(A_j = \begin{pmatrix} -2 \\ 0 \end{pmatrix}\), \(A_v = \begin{pmatrix} -8 \\ 6 \end{pmatrix}\), and \(A_w = \begin{pmatrix} -6 \\ 8 \end{pmatrix}\).

b. Best draw all eight vectors based at the origin. Easy.

c. A rotates any vector counterclockwise by \(\frac{\pi}{2}\) and doubles its magnitude.

4d. The determinant \(A_i = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\), of \(A\) is \(\det A = 0 \cdot 0 - (-2) \cdot 2 = 4\).
Show your work - explain what you are doing. It is YOUR responsibility to demonstrate that you have mastered the material of this class.

1. Consider the contour diagram of a function \( z = f(x,y) \) given on the right. Suppose that the spacing between the contours is 100, and that the minimum is at \( z = -800 \), light tones indicating large values of \( f(x,y) \) and dark tones small values of \( f(x,y) \).
   a.) Sketch the graph of the cross-section of \( z = f(x,0) \).
   b.) List the approximate coordinates of all saddle points.

<table>
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<th>t</th>
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<th>4</th>
<th>6</th>
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<td>50</td>
<td>40</td>
<td>34</td>
<td>30</td>
<td>30</td>
<td>34</td>
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</tbody>
</table>

2. Consider the table on the left giving position data for a parameterized curve \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \).
   a. Estimate the velocity and speed at \( t=7 \) - explain what you are doing, and justify your approach.
   b. Estimate the acceleration at \( t=7 \) - explain what you are doing, and justify your approach.

3. Calculate the length of the parameterized curve \( \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} \) with \( 0 \leq t \leq 4\pi \). Demonstrate that your answer is reasonable.

4. Suppose that a particle moves along a curve at constant speed. Differentiate the square of the speed \( \mathbf{v}(t) \cdot \mathbf{v}(t) \) to explain why the acceleration is perpendicular to the velocity.

5. Consider the function \( f(x,y) = \sqrt{x^2 + y^2} \).
   a.) Describe the shape of the graph in words, and sketch the graph.
   b.) Sketch the cross-sections of the graph parallel to the xz-plane, e.g. for \( y = -1, 1, 2 \).
   c.) Sketch and describe in words the contours for \( f(x,y) = 0, 1, 2 \).

6.a. Find the three points where the plane \( 2x+3y+6z=12 \) intersects the three coordinate axes.
   b. Find the lengths of the three sides of the triangle with corners at the points calculated above.
   c. Calculate the area of this triangle.
Always justify your work and explain your reasoning. You are encouraged to use calculators or the computers for calculations, but you may not use the Internet-connections to the outside world. Some problems may require numerical methods as closed form formulas may not be possible.

1. Consider the triangle \( ABC \) with vertices \( A(8.9,1.2,0.0) \), \( B(1.2,8.9,0.0) \) and \( C(0.0,1.2,6.7) \).
   a.) Find the length \( c \) of the edge \( AB \).
   b.) Calculate the angle between the line segments \( AB \) and \( AC \).
   c.) Calculate the area of the triangle.
   d.) Find a \underline{unit-vector} \( \vec{N} \) that is perpendicular to the triangle.
   e.) Find an equation for the plane that contains the triangle.
   f.) Find the direction of the line of intersection of this plane with the \( xy \)-plane.
   g.) Find a parametric equation for this line of intersection.

2. Given that \( \vec{u} \), \( \vec{v} \), and \( \vec{w} \) are vectors in 3-space, and that \( t \) is a scalar, decide which of the following are scalars, which are vectors, and which quantities are not defined? \underline{Justify your reasoning!}
   
   a. \( (\vec{u} \times \vec{v}) \times \vec{u} \)  
   b. \( t + \vec{w} \)  
   c. \( \vec{v} + t(\vec{u} \times \vec{w}) \)  
   d. \( \frac{\vec{v} + \vec{w}}{\vec{v} - \vec{w}} \)  
   e. \( \frac{\vec{v} + \vec{w}}{\vec{v} \cdot \vec{w}} \)  
   f. \( \frac{\vec{u} \times \vec{w}}{\vec{v} \times \vec{w}} \)

3. Consider the parameterized curve \( \vec{r}(t) = t \cos(t)\vec{i} + t \sin(t)\vec{j} \) with \( 0 \leq t \leq 4 \).
   a. Describe the curve in words. Sketch the curve.
   b. How many times does the curve cross the positive \( x \)-axis? \( \textbf{Hint:} \) How large is \( e^4 \)?)
   c. Calculate the velocity and the speed as functions of time. (Chain and product rule!)
   d. Sketch the graph of the speed – label the axes! Is the speed increasing, decreasing or constant?
   e. How large is the speed at \( t = 3 \)? Compare this with the circumference of a circle with radius 3.
   f. Calculate the length of the curve.
   g. Find the \textit{smallest time} \( t_1 \) at which the velocity vector \( \vec{v}(t_1) \) is horizontal and points to the left.

4. Suppose that a parameterized curve is such that at all times the velocity vector \( \vec{v}(t) \) and the acceleration vector \( \vec{a}(t) \) are perpendicular to each other. Use the following string of equations
   
   \[
   \frac{d}{dt} \|\vec{v}(t)\|^2 = \frac{d}{dt} (\vec{v}(t) \cdot \vec{v}(t)) = 2\vec{v}(t) \cdot \vec{a}(t)
   \]
   
   to explain (!) what this tells you about the speed \( \|\vec{v}(t)\| \).