1.a.) Calculate all 1st and 2nd order partial derivatives of \( f(x, y) = \ln(xy) \).

b.) For which \((x, y)\) is \(\ln(xy)\) defined?

c.) Sketch the level-curves \( f(x, y) = c, \ c = -1, 0, 1, 2. \) Find equations for these curves in the form \( y = \ldots \)

d.) Find the equation of the tangent plane of \( z = \ln(xy) \) at \((x, y) = (1, 1)\).

2.) A ball thrown from ground level (with initial speed \(v\) and at an angle \(\alpha\) with the horizontal) hits the ground at a distance \( s = \frac{v^2\sin(2\alpha)}{g} \) \((g \approx 10m/s^2)\) is gravitational acceleration.

a.) Find a reasonable domain for the values of \(\alpha\) and \(v\). (Hint: A football field is about \(s = 100m\) long.)

b.) Describe the shape of the graph, and/or sketch the graph of \(s\) as a function of \((v, \alpha)\).

c.) Calculate the differential \(ds\). What does the sign of \(\frac{ds}{d\alpha}(20, \frac{\pi}{3})\) tell you?

d.) Use the linearization of \(s\) about \((v, \alpha) = (20, \frac{\pi}{3})\) to estimate how \(\alpha\) should change to get approximately the same distance \(s\) when \(v\) changes to \(19\frac{9}{2}\).

3.) Let \(r = f(x, y) = \sqrt{x^2 + y^2}\) and \(\Theta = \tan^{-1}(\frac{y}{x})\) denote the change from rectangular to polar coordinates.

a.) Explicitly express the differential \(dr\) in terms of \(dx\), \(dy\) and the partial derivatives of \(f\).

b.) Describe/sketch the graph of \(r = f(x, y)\).

c.) Find \(r\) when \((x, y) = (6, 8)\). Suppose \(x = 6\) and \(y = 8\) have been measured with 1% and 2% relative errors. How large are the corresponding maximal absolute errors \(\Delta x\) and \(\Delta y\)? Using differentials, estimate the resulting absolute and percentage errors for \(r = \sqrt{x^2 + y^2}\).

d.) At time \(t\) a vehicle is at \(x = 3t - 1\), and \(y = 1 + \frac{t}{2}\). Compute \(\frac{dr}{dt}\) (first find \(dx\), \(dy\), then \(dr\) in terms of \(dt\)).

e.) **Bonus question:** Overlay the curve \((x, y) = (3t - 1, 1 + \frac{t}{2})\) with the contours of \(r = f(x, y)\). For which \(t\) does \(r = F(t)\) attain its minimum? Explain (i) what happens in the contour diagram, and (ii) what is \(dr\) at this time; (iii) sketch the graph of \(r = F(t)\).

4.) In the contour plot of a **continuous** function \(f(x, y)\) which of the following can never happen: a contour stops suddenly; two contours cross each other, merge, or separate. Justify your answer! (Distinguish the cases where two contours correspond to the same or to different function values.) For inspiration consider the following **fake-contour** plot:

![Contour Plot](image)

**Bonus question:** Find formulas for functions whose contours exhibit any of the above (crossing, stopping, merging)?
Test 2

Sample Solutions

1. a) \( f(x,y) = \ln(xy) \), \( f_x = \frac{1}{x}, \ f_y = \frac{1}{y}, \ f_{xx} = -\frac{1}{x^2}, \ f_{xy} = f_{yx} = 0, \ f_{yy} = -\frac{1}{y^2} \)

b) \( xy \) must be \( > 0 \), i.e. \( xy \) must have same sign, see picture.

c) \( \ln(xy) = c \Leftrightarrow xy = e^c, \ y = (e^c)^{\frac{1}{x}} \)

d) \( f(1,1) = 0, \ f_x(1,1) = 1 = f_y(1,1) \Rightarrow z = (x-1) + (y-1), z = x + y - 2. \)

2a) If ball is not thrown backwards \( 0 \leq \alpha \leq \frac{\pi}{2} \).

Try e.g. \( \alpha = \frac{\pi}{4} \), solve \( 100 = s = \frac{\sqrt{2}}{16} \) for \( v = \frac{\sqrt{1000}}{\sqrt{8}} \approx 30 \)

e.g. \( 0 \leq v \leq 50 \) mph looks reasonable (\( \approx 110 \) mph)

2b) \( \alpha = \text{const.} \Rightarrow s = \text{const.} \frac{v^2}{g} \)

\( v = \text{const.} \Rightarrow s = \text{const.} (\sin \alpha) \)

c) \( ds = 2v \sin(2\alpha) \, dv + 2v \cos(2\alpha) \, dx \), \( \frac{ds}{dx} = (20, \frac{\pi}{3}) = 80 \cos(\frac{2\pi}{3}) < 0 \)

\( \Rightarrow \) increasing angle further reduces the distance

d) Want \( ds = 0 \), given \( v = 20, \alpha = \frac{\pi}{3} \), \( dv = -1 \) solve for \( dx = \frac{\tan(2\alpha)}{v} \)

about \( \frac{\pi}{6} \) degrees

3a) \( dr = \frac{2x}{2v} \, dx + \frac{2y}{2v} \, dy (\text{abusive mixed notation, } x, y \text{ and } t \text{ mixed}) \)

b) right circular cone with vertex at \((0,0,0)\), opening along \(-x, -y, 1\).

\[ \Delta t \approx dr = \frac{4}{10} \] \( 0.06 + \frac{3}{10} \) \( 0.16 = 0.16 \approx 1.64 \% \)

d) \( dx = 3 \, dt, \ dy = -\frac{4}{3} \, dt, \ dr = \left(3 \frac{1}{3} - \frac{4}{3} \cdot \frac{4}{13}\right) dt \)

\( \frac{1}{\sqrt{(3t-1)^2 + (1+\frac{4}{3})^2}} \) \( (9t-1-\frac{16}{3}) \) \( \frac{dt}{(3t-1)^2 + (1+\frac{4}{3})^2} \)

(\( \text{for bonus question see reverse side} \))

4) If \( f(x,y) \) is continuous then two contours corresponding to different values \( c_1 \) and \( c_2 \) cannot cross, touch, merge, split... At any such point \( f(x,y) \) would have to have 2 function values.

As \( f(x,y) = xy \) with \( c = 0 \) shows \( \frac{x}{t} \) contours with same value \( c \) can cross (a saddle in this case).

A single contour can suddenly stop, but only if it is so special that the function carries greater values on both sides (or smaller on both sides).

Example: \( f(x,y) = x^2 + |x| + y^2 \).

Graph looks like \( y = \frac{x^2}{2} \) but shifted up, left and stretched.

Bonus question: Eliminating \( t \) yields \( t = \frac{(x+1)}{3} \) and \( \gamma = \frac{12}{(x+1)} + 1. \)

Either play \( w \) plot to find minimum of \( \gamma(t) = 5 \) at \( t = \frac{1}{3} \) or solve \( \frac{d\gamma}{dt} = 0, \text{i.e.} \ 9t^4 + t^3 - 4t - 16 = 0 \)

via trial/error, Newton, MAPLE.

Here \( dt = 0 \) and curve/contours are tangent.
1.) Find all critical points of the function \( f(x, y) = 2x^3 - 9x^2 + 12x + y^3 - 3y^2 - 3y + 5 \), and determine which of these are local maxima, local minima, saddle points and which are neither of these.

2.a.) Find all 1st and 2nd-order partial derivatives of the function \( f(x, y) = \frac{1}{xy} \).

b.) In which direction is \( f(x, y) \) increasing at the greatest rate at the point \( (x, y) = (-2, 3) \)?

c.) What is the equation and what is the slope of the contour of \( f(x, y) \) passing through \( (x, y) = (-2, 3) \)?

d.) Estimate \( f(-2.01, 2.98) \) without using a calculator. Explain.

3.a) Find the values of \( a \) and \( b \) that minimize the least squares distance of the function \( y = a/x + b/x^2 \) from the points (1,10), (2,5), and (5,1). *(The problem on the test will require less complicated algebraic calculations.)*

b.) If instead the function is of the form \( y = a/x + b/x^2 + c/x^3 \), what is the minimal least squares distance? What makes this problem so different from the problem in part a.)?

4.) Let \( r = f(x, y) = \sqrt{x^2 + y^2} \) and \( \Theta = \tan^{-1}(\frac{y}{x}) \) denote the change from rectangular to polar coordinates.

a.) Explicitly express the differential \( dr \) in terms of \( dx \), \( dy \) and the partial derivatives of \( f \).

b.) Describe/sketch the graph of \( r = f(x, y) \).

c.) Find \( r \) when \( (x, y) = (6, 8) \). Suppose \( x = 6 \) and \( y = 8 \) have been measured within 1% and 2% relative errors. How large are the corresponding maximal absolute errors \( \Delta x \) and \( \Delta y \)? Using differentials, estimate the resulting absolute and percentage errors for \( r = \sqrt{x^2 + y^2} \).

5.) Give simple explicit examples of quadratic functions of two variables (i.e. of the form \( f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f \)) whose contours are a.) circles, b.) ellipses that are not circles, c.) hyperbolas, and d.) parallel lines.

6.) Given a contour diagram, estimate the direction and magnitude of the gradient vector at any point, estimate the sign of any first and second order partial derivative, and estimate the value of any directional derivative. (Work with a class-mate, each draws a contour diagram on graph paper, labelling the contours, and marking one or several points. Exchange the pictures, work the above exercises - discuss the results; and if applicable, discuss why (complain that) some problems can't be done with the given information.)

7.a.) Review the handout M-W 9/27-29 and quiz 4. Be able to match algebraic expressions and features of the picture. (Quiz your study partner!)

b.) At a fixed point \( (a, b, f(a, b)) \) all tangent lines to the graph of \( z = f(x, y) \) lie in the same plane, called the tangent plane. What arguments do you know to convince a skeptical class-mate that this is indeed true? (pictures, algebraic-analytic proofs, anything that convinces ...)

For bonus credit: Find the maximum value of the function \( f(x, y) = \int_x^y \sin t \, dt \).
1.a.) Sketch the typical contours near a local maximum, near a local minimum, and near a saddle point.
b.) Find all critical points of the function \( f(x, y) = x^3 - 9x + 2y^2 - 6y^2 - 18y - 5 \), and determine which of these are local maxima, local minima, saddle points, and which are neither of these.

2.) A ball thrown from height \( h \) with initial velocity \( v \) at an angle \( \alpha \) (with the horizontal) after \( t \) is at the height

\[ z = h + vt \sin \alpha - \frac{1}{2} gt^2 \]

(where \( g = -9.81 \frac{m}{s^2} \)).

a.) Calculate all four first order partial derivatives of \( z \).
Is \( \frac{\partial z}{\partial x} \) always positive? What is the practical meaning of this?
b.) Considering \( h = 2 \) (meters) and \( v = 20 \) (meters per second) fixed, find the maximum value of \( z \) as a function of \( t \) and \( \alpha \) (use \( g \approx 10 \) for computational convenience).

3.) Consider the picture given on the right.
a.) Find formulas for the height of the point \( G \), the slope of the secant line through \( AG \), and the tangent line (in the \( x \)-direction) through the point \( G \).
b.) If \( dx = |AB| \) and \( dy = |AE| \), mark the following on the picture: \( dx, \Delta x, \) and lines whose slopes are \( f(a + \Delta x, b + \Delta y) - f(a, b) \)/\( h \) and \( (D_{a,b}f)(a, b) \), where \( \vec{u} \) is a unit vector in the direction of \( \vec{w} = (\Delta x)\hat{i} + (\Delta y)\hat{j} \) and \( h \) is the length of \( \vec{w} \).
c.) If \( z = x \sin y \), \( (a, b) = (10, \pi/6) \), \( dx = 0.08 \), \( dy = 0.06 \) find \( dz \).

For bonus credit: Find formulas for the vectors \( \vec{AB}, \vec{AF}, \) and for the area of the parallelogram \( ACIF \).

4.) Consider the contour diagram given on the left.
a.) Find all saddle points.
b.) Show the directions of the gradients at \( P \) and \( Q \).
c.) Which of \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \) are positive at \( P \)?
d.) Estimate \( (D_{a,b}f)(Q) \) where \( \vec{u} = (\hat{i} + \hat{j})/\sqrt{2} \). (Contour values as indicated, 1 box = 1 unit.)

5.a) Find the function \( y = a + bx \) that minimizes the least squares distance from the points \((-2, 9), (0, 3), \) and \((1, 0)\).
b.) What is the least squares distance of this line from the given points? What does this mean geometrically?

For bonus credit: Find the maximum value of the function \( f(x, y) = \int_a^b \sin t \, dt \).
1.a.) Local minimum
Local maximum
Saddle point

The partial derivatives are \( f_x(x, y) = 3a^2 - 9 \) and \( f_y(x, y) = 6y^2 - 12y - 18 = 6(y^2 - 3)(y + 1) \). These are both zero when \((x, y)\) is any of the following: \( A = (-\sqrt{3}, -1) \), \( B = (-\sqrt{3}, 3) \), \( C = (\sqrt{3}, -1) \), \( D = (\sqrt{3}, 3) \).

The second partial derivatives are \( f_{xx}(x, y) = 6a \), \( f_{xy}(x, y) = 0 \) and \( f_{yy}(x, y) = 12(y - 1) \). The discriminant \( D = f_{xx}f_{yy} - f_{xy}^2 = 72(y - 1) \) is negative at \( B \) and \( C \) which therefore are saddle points, and is positive at \( A \) and \( D \). Since \( f_{xx} \) is negative at \( A \) and positive at \( D \), there is a local maximum at \( A \) and a local minimum at \( D \).

2.a.) If \( z = h + vt \sin \alpha - \frac{1}{2}gt^2 \), then

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial y} = 1, \quad \frac{\partial z}{\partial y} = t \sin \alpha, \quad \frac{\partial z}{\partial t} = v \sin \alpha - gt, \quad \frac{\partial z}{\partial \alpha} = vz \cos \alpha
\]

As long as time \( t \) is positive and the angle \( \alpha \) is between 0 and \( \pi \), \( \frac{\partial z}{\partial \alpha} \) is positive. Practically, this means that increasing the velocity will increase the height (at any given time, etc.)

b.) Fixing \( h = 2 \) and \( v = 20 \) yields \( \frac{\partial z}{\partial t}(t, \alpha) = 20 \sin \alpha - 10t \) and \( \frac{\partial z}{\partial \alpha}(t, \alpha) = 20t \cos \alpha \). Setting these equal to zero, the only meaningful solutions are \( \alpha = \frac{\pi}{2} \) (i.e. vertically up), and \( t = 2 \) (i.e. the maximum height is attained 2 seconds after the ball is thrown). At this time the height is \( z_{\text{max}} = 2 + 20 \cdot 2 \cdot 1 - \frac{1}{2} \cdot 10 \cdot 2^2 = 22 \) meters above ground.

3.) The height of the point \( G \) is \( f(a, b + \Delta y) \), the slope of the secant line through \( A \) and \( G \) is \( \frac{f(a, b + \Delta y) - f(a, b)}{\Delta y} \), and the slope of the tangent line (in the \( x \)-direction) through the point \( G \) is \( \frac{\partial f}{\partial a}(a, b + \Delta y) \).

b.) \( dx \) and \( \Delta x \) are the increments \( |HI| \) and \( |HL| \), respectively. The lines through the points \( A \) and \( I \), and through \( A \) and \( L \) have the slopes \( \left(f(a + \Delta x, b + \Delta y) - f(a, b)\right)/h \) and \( \left(D_{a}f\right)(a, b) \), respectively.

c.) With the given data \( f_x(x, y) = \sin y, f_y(x, y) = \cos x, \) thus \( f_x(10, \pi/6) = 0.5, f_y(10, \pi/6) = 5\sqrt{3} \), and thus \( dx = f_x(a, b)(\Delta x) + f_y(a, b)(\Delta y) = 0.5 \cdot 0.08 + 5\sqrt{3} \cdot 0.06 = 0.04 + 0.3 \cdot \sqrt{3} \approx 0.5596 \).

For **bonus credit**: \( \overrightarrow{AC} = (\Delta x)(\overrightarrow{i} + f_x(a, b)\overrightarrow{k}), \overrightarrow{AP} = (\Delta y)(\overrightarrow{i} + f_y(a, b)\overrightarrow{k}) \), and thus the area of the parallelogram \( ACIF \) equals the magnitude of \( \overrightarrow{AC} \times \overrightarrow{AP} \), which is \( \sqrt{1 + f_x^2(a, b)^2 + f_y^2(a, b)^2} \cdot |\Delta x| \cdot |\Delta y| \).

4.a.) Saddle points as marked (contours cross, and nearby contours resemble hyperbolas). When putting \((0, 0)\) at the bottom left, there are saddles near \((1, 1)\), and \((5, 6)\).

b.) As shown, perpendicular to the contours, and in the direction of increasing function values approximately in the direction \( \overrightarrow{i} + \overrightarrow{j}/\sqrt{2} \) both at \( P \) and at \( Q \).

c.) At the point \( P \) \( f \) is increasing at an increasing rate in the \( x \)-direction, is increasing at an increasing rate in the \( y \)-direction, and the rate of increase in the \( x \) direction also increases as \( y \) increases. Therefore at \( P \) the following are all positive: \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \).

c.) The vector \( \overrightarrow{u} \) points diagonally up to the right. In this direction the function increases (near \( Q \)) approximately by \( 20 \) units when \( x \) and \( y \) increase by one unit each. Therefore the directional derivative equals approximately \( 20/\sqrt{2} \approx 14.14 \).

5.) The "square distance" \( s \) from the given points is

\[
s = (9 - (-2m + b))^2 + (3 - (0m + b))^2 + (0 - (1m + b))^2 = (9 + 2m - b)^2 + (3 - b)^2 + (m + b)^2
\]

The partial derivatives are

\[
\frac{\partial s}{\partial m} = 2(9+2m-b)\cdot 2+0+2(m+b) = 10m-2b+36 \quad \text{and} \quad \frac{\partial s}{\partial b} = 2(9+2m-b)(-1)+2(3-b)(-1)+2(m+b) = -2m+6b-24
\]

Equating these to zero yields \( m = -3 \) and \( b = 3 \). In this case \( s = s(-3, 3) = 0 \) which means that the three points actually all lie on one line, namely on \( y = 3(1-x) \).

**Bonus credit**: Review the fundamental theorem from calculus I, use the second derivative test from calculus III, recall the area-interpretation of the definite integral; be prepared for it again on the next exam.
Vectors and parametrized curves

1.) Draw the picture of a racetrack (top-view). At approximately 8 to 10 positions mark typical velocity and acceleration vectors.
Select three interesting points and explain in words what is happening here.

You pick the level of difficulty so that you can demonstrate your understanding; diversity of situations and overall consistency are the most important criteria.

2.) Consider the parametrized curve \( \mathbf{r}(t) = (13 \cos t) \mathbf{i} + (5 \sin t) \mathbf{j} \) for \( 0 \leq t \leq 2\pi \).
a.) Describe the curve in words. If possible find an implicit and/or explicit description of the curve.
b.) Calculate the velocity \( \mathbf{v}(t) \) and the acceleration \( \mathbf{a}(t) \).
c.) Find the speed \( |\mathbf{v}(t)| \) and sketch its graph.
d.) Using parts b.) and c.), describe the general features of the parallel \( a_1(t) \) and perpendicular \( a_2(t) \) components of the acceleration? (E.g. zero, positive, negative, constant, increasing or decreasing . . .)
(Alternatively find formulas and sketch their graphs).
e.) Calculate the length of the curve.
f.) (for bonus credit) Calculate a formula for the curvature, and sketch its graph.

3.) Consider one of the following lists of equations:

\[
\frac{d}{dt} |\mathbf{v}(t)|^2 = \frac{d}{dt} (\mathbf{v}(t) \cdot \mathbf{v}(t)) = \mathbf{a}(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{a}(t) = 2 \mathbf{a}(t) \cdot \mathbf{v}(t)
\]

\[
|\mathbf{v}|^2 = |\mathbf{b} - \mathbf{a}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2 \mathbf{a} \cdot \mathbf{b}
\]

Select one case, and (i) explain what this proves (when suitably interpreted), and
(ii) justify why one can perform each step.

Hints: The first equation may be read in either direction, after appending \( = 0 \) at the end, or adding \( = 0 \) to the beginning. In the second case think triangles and again perpendicular (or right angle).

4.) Let \( T \) be the triangle with corners \( A(-2, 4, 5), B(2, 7, -3), \) and \( C(9, -1, 4) \).
a.) Find a vector perpendicular to the triangle.
Calculate the area of \( T \).
Find a numerical value for the interior angle \( \alpha \) at the point \( A \).
b.) Write down the equation of the plane \( P \) containing \( T \).
d.) Does the point \( D(1, 1, 1) \) lie in the plane? Justify your answer.
If not find the distance of \( D \) from the plane \( P \) (consider the dotproduct of the normal to \( P \) and \( \overrightarrow{AD} \)).
1.) At point A the track is straight, and the acceleration is parallel to the velocity. The car is speeding up, so \( \vec{a} \) and \( \vec{v} \) point in the same direction. At both B and C the track is curved, and the perpendicular component of the acceleration cannot be zero - it points towards the inside of the curve. At B the car is slowing down for the turn, and at C it is again speeding up, and the acceleration vector makes an angle of larger than \( \pi/2 \) at B and smaller than \( \pi/2 \) at C with the velocity vector.

2.a.) \( \vec{r}(t) = (13\cos t) \hat{i} + (5\sin t) \hat{j} \) is a parametrization of an ellipse, centered at the origin, with half axes of length 13 and 5, traversed counterclockwise. To obtain an implicit/explicit description eliminate the parameter \( t \) from the equations \( x = 12\cos t \) and \( y = 5\sin t \): \( (\frac{x}{13})^2 + (\frac{y}{5})^2 = \cos^2 t + \sin^2 t = 1 \), or explicitly \( y = \pm 5\sqrt{1 - (\frac{x}{13})^2} \).

b.) Straightforward differentiations yield \( \vec{v}(t) = (-13\sin t) \hat{i} + (5\cos t) \hat{j} \), and \( \vec{a}(t) = (-13\cos t) \hat{i} + (-5\sin t) \hat{j} \).

c.) The speed is \( |\vec{v}(t)| = \sqrt{169\sin^2 t + 25\cos^2 t} = \sqrt{25 + (12\sin(t))^2} \). The graph on the right is obtained using the following commands in MAPLE: \( r:=(13\cos(t),5\sin(t)); v:=diff(r,t); p:=plot(sqrt(dotprod(v,v)),t=0..2*Pi); \)

It shows a slowing down at the points far from the center, and a speed-up at points near the center.

d.) The speed is small at the points furthest from the origin and large near the center; consequently the parallel acceleration will change sign on the coordinate axes, and have the same sign as \( xy \). The perpendicular component of the acceleration is never zero as the curve is convex (always turning left).

Formulas are \( a_{\|}(t) = (\vec{a}(t) \cdot \vec{v}(t))/|\vec{v}(t)| = \frac{72\sin(2t)}{\sqrt{25 + (12\sin(t))^2}} \) and \( a_{\perp}(t) = |\vec{a}(t) \times \vec{v}(t)|/|\vec{v}(t)| = 65/\sqrt{25 + (12\sin(t))^2} \).

e.) The length of the curve is given as the integral of the speed. This is an elliptic integral that is easiest evaluated numerically \( L = \int_0^{2\pi} |\vec{v}(t)| dt \approx 59.38 \) (using the integration and evalf commands in MAPLE).

f.) The curvature is given by \( \kappa(t) = |\vec{a}(t) \times \vec{v}(t)|/|\vec{v}(t)|^3 \), i.e. very similar to the expression for \( a_{\perp}(t) \), and easily graphed in MAPLE. The tight turns on the \( x \)-axes at \( t = \pi, 2\pi \) show up clearly as maxima of the curvature function.

3.) The first sequence of equations shows that the speed is constant if and only if the acceleration is perpendicular to the acceleration at all times. Specifically, if the speed is constant, then so is its square. The derivative of a constant is zero. On the other hand, the magnitude squared of a vector equals the dot product of the vector with itself. Dot-products may be differentiated using a product rule just like scalar functions. The last step uses that dot products are commutative. Finally, a dot product is zero if and only if the vectors are perpendicular.

The second sequence of equations is essentially a proof of Pythagoras’ theorem: Let \( \vec{a}, \vec{b}, \vec{c} = \vec{b} - \vec{a} \) be three sides of a triangle. Again use magnitude of a vector squared equals the dot product with itself. Dot-products are distributive and commutative. Thus \( |\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \) if and only if \( \vec{a} \cdot \vec{b} = 0 \), i.e. if and only if the triangle is a right triangle.

4.) Write \( \vec{v} = \vec{AB} = 4\hat{i} + 3\hat{j} - 8\hat{k} \) and \( \vec{w} = \vec{AC} = 11\hat{i} + 5\hat{j} - 9\hat{k} \).

The crossproduct \( \vec{N} = \vec{v} \times \vec{w} = 43\hat{i} - 84\hat{j} - 53\hat{k} \) using MAPLE) is perpendicular to the triangle.

The area equals half the magnitude of \( \vec{N} \) i.e. area \( = \frac{1}{2} |\vec{N}| = \sqrt{17114} \approx 108.23 \).

Using the definition of the dot product \( a = \cos^{-1}((\vec{v} \cdot \vec{w})/(|\vec{v}||\vec{w}|)) \approx 1.241 \) (this is approximately 71.13 degrees).

b.) Using the notation \( \vec{R} = x\hat{i} + y\hat{j} + z\hat{k} \) one equation of the plane is \( \vec{N} \cdot \vec{R} = \vec{N} \cdot 0\hat{A} \), i.e. \( 43x + 84y + 53z = 515 \).

c.) Evaluating \( \vec{N} \cdot \vec{R} \) with \( \vec{R} = \hat{i} + \hat{j} + \hat{k} \) yields \( \vec{N} \cdot \vec{R} = 180 \) which does not equal the right hand side of above equation of the plane.

The distance of \( D \) from the plane equals e.g. the component of the vector \( \vec{AD} \) parallel to the vector \( \vec{N} \).

If \( \gamma \) denotes the angle between these vectors then we want the value of \( |\vec{AD}| \cos \gamma \), which is easiest calculated using dot products:

\[ |\vec{AP}| \cos \gamma = \frac{\vec{AP} \cdot \vec{N}}{|\vec{N}|} \approx 3.095 \]
Show your work - explain what you are doing. It is YOUR responsibility to demonstrate that you have mastered the material of this class.

You may bring in one page of formulas - which you have to turn in with the test.

1.a. Find all critical points of the function \( f(x, y) = x^3 - 3x^2 - y^3 + 3y \).
b. Determine at which a local maximum, a local minimum or a saddle point occurs.
c. Show all your work, and explain your reasoning step by step!

2. Consider the function \( f(x, y) = xy^2 \).
a. Sketch several cross-sections of the graph parallel to the \( yz \)-plane. Explain.
b. Sketch several cross-sections of the graph parallel to the \( xz \)-plane. Explain.
c. Describe the shape of the graph in words.
d. Find the equation of the tangent plane to the graph of \( z = f(x, y) \) at \( (x, y) = (0, 0) \).
e. Find the second order Taylor approximation for \( f(x, y) \) at \( (x, y) = (0, 0) \).

3. Consider the table of function values for a function \( z = f(x, y) \) on the left. (\( x \)-values listed across, \( y \)-values up-down.)
a. Sketch a contour diagram.
b. Describe the shape of the graph in words.
c. Sketch the graphs for cross-sections parallel to the \( xz \)-plane for various values of \( y \).
d. Estimate the following derivatives:
\[
\frac{\partial f}{\partial x}(1, 2), \quad \frac{\partial f}{\partial y}(1, 2), \quad \frac{\partial^2 f}{\partial x \partial y}(1, 2)
\]
e. Is \( \frac{\partial^2 f}{\partial x \partial y}(1, 2) \) positive, negative or zero? Explain why! Is \( \frac{\partial^2 f}{\partial x^2}(1, 2) \) positive, negative or zero? Explain why!

4.a. Calculate the curvature \( \kappa(t) \) for the curve \( (x, y) = (t \cos(t), t \sin(t)) \) as a function of time.
b. Sketch the graph of the curvature as a function of time for \( t > 0 \).
c. Comment in one or two sentences how this graph agrees with your expectations.

5. For the "coordinate change" \( x = r \cos(\Theta), y = r \sin(\Theta) \), with "inverse" \( r = \sqrt{x^2 + y^2}, \Theta = \arctan(y/x) \),
a. calculate the total derivatives, that is, the matrices of partial derivatives
\[
A = \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \Theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \Theta}
\end{pmatrix}
\]  
and
\[
B = \begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \Theta}{\partial x} & \frac{\partial \Theta}{\partial y}
\end{pmatrix}
\]
b. Calculate the determinant of the matrix \( A \) (simplify the result).
c. Is it true that \( \frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial x} = 1 \)?

Bonus: Calculate the determinant of \( B \) (liberally mix \((x, y)\) and \((r, \Theta)\) coordinates!)

Calculate and simplify the matrix products \( AB \) and \( BA \).
Show your work - explain what you are doing. It is YOUR responsibility to demonstrate that you have mastered the material of this class.

1. Consider the function \( r = \sqrt{x^2 + y^2} \).
   a. Calculate the partial derivatives \( r_x, r_y, r_{xx}, r_{xy}, r_{yy} \).
   b. Find the equation of the tangent plane (to the graph of \( r(x,y) \)) at the point \((a,b)=(8,6)\). 
   c. Use your result from b. to estimate the value of \( r \) when \((x,y)=(8.1, 5.8)\), and compare with the exact value.

2. Consider the illustration on the right.
   a. Find an expression (in terms of the quantities listed) for the slope of the line-segment connecting the points \( A \) and \( L \).
   b. Find a point in the illustration whose \( z \)-coordinate equals 
   \[ f(a,b) + f_u(a,b) \Delta x + f_v(a,b) \Delta y. \]
   c. Where in the illustration can you find the directional derivative \( D_u f(a,b) \) 
   (when \( u \) is a unit vector pointing in the direction of \( \Delta x \mathbf{i} + \Delta y \mathbf{j} \)?)

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-4</td>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>-4</td>
</tr>
</tbody>
</table>

3. Consider the table of values of a function \( f(x,y) \) shown on the left (\( x \) listed left to right, \( y \) listed from bottom to top).
   a. Is \( f \) a linear function? Justify your answer?
   b. Estimate the gradient of \( f(x,y) \) at the points \((-2,1) \) and \((0,0)\). Explain what you are doing.
   c. Give the equation of the tangent plane (to the graph of \( z=f(x,y) \)) at the point \((0,0,0)\).

4. Consider the contour diagram shown on the right.
   a. Find all saddle points.
   b. Show the directions of the gradients at the points \( P \) and \( Q \).
   c. Which of \( f_x, f_y, f_u, f_v \) are positive at the point \( P \)?
   d. Estimate the directional derivative \( D_u f(Q) \) 
   where \( u \) is a unit vector pointing in the direction of \( \mathbf{i} + \mathbf{j} \). 
   (The width and height of each box is 1 unit length.)

5. a. State the special case of the chain rule for the case that 
    \( z=f(x,y) \) is a function of two variables and \((x,y)=(g(t),h(t)) \) is a 
    parameterized curve – i.e. write the derivative \( z_t \) in terms of 
    \( z_x, z_y, x_t = x' \) and \( y_t = y' \).
    b. Rewrite your answer from a. in terms of gradients and 
    velocity vectors – precisely state what your symbols denote.
    c. In the special case of \( z=1/(1+x^2+y^2) \), \( x=t, y=(1-t^2) \) find \( z_t(1) \).
1. Consider the function \( f(x, y) = x^3 - y^3 + 3y^2 - 3x - 7 \).

You may earn up to 10 BONUS points if you work this problem for \( g(x, y) = 171x^3 + 84x^2y - 63xy^2 + 172y^3 - 5625x - 7500y \) instead of the \( f(x, y) \) above.

Note, this alternative has nice solutions, but almost requires a computer.

a. Find all critical points of \( f(x, y) \).
b. Classify the critical points using second derivatives.

2. Consider the function \( z = \sqrt{x^2 + y^2} \).

a. Describe the shape of the graph in words.
b. Sketch vertical cross-sections of the graph, parallel to the \( xz \) plane for \( y = -2, -1, 0, 1, 2 \).

You may want to overlay these on the same set of axes.
c. Sketch a contour diagram of \( z \) and mark the gradient vectors at about a dozen different points.

BONUS: Use Lagrange multipliers to find the minimum of \( z \) subject to the constraint \( 3x + 4y = 25 \).

3. The table on the right shows the monthly payment \( P = f(N, r) \) (in $) for a $10,000 loan with an annual interest rate of \( r \%) \) to be paid off over \( N \) months.

<table>
<thead>
<tr>
<th>( N/r )</th>
<th>8.50</th>
<th>8.75</th>
<th>9.00</th>
<th>9.25</th>
<th>9.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>205.17</td>
<td>206.37</td>
<td>207.58</td>
<td>208.80</td>
<td>210.02</td>
</tr>
<tr>
<td>54</td>
<td>223.50</td>
<td>224.70</td>
<td>225.90</td>
<td>227.10</td>
<td>228.30</td>
</tr>
<tr>
<td>48</td>
<td>246.48</td>
<td>247.67</td>
<td>248.85</td>
<td>250.04</td>
<td>251.23</td>
</tr>
<tr>
<td>42</td>
<td>276.10</td>
<td>277.27</td>
<td>278.45</td>
<td>279.62</td>
<td>280.80</td>
</tr>
<tr>
<td>36</td>
<td>315.68</td>
<td>316.84</td>
<td>318.00</td>
<td>319.16</td>
<td>320.33</td>
</tr>
</tbody>
</table>

a. Is \( f \) a linear function? Explain why or why not.
b. What are the units of \( \frac{\partial P}{\partial N} (48, 9) \) and \( \frac{\partial P}{\partial r} (48, 9) \)?
c. Estimate the values of \( \frac{\partial P}{\partial N} (48, 9) \) and \( \frac{\partial P}{\partial r} (48, 9) \).
d. Explain the practical meanings of \( \frac{\partial P}{\partial N} (48, 9) \) and \( \frac{\partial P}{\partial r} (48, 9) \), and the significance of their signs (+/-).
e. Find a formula for the local linearization \( L(N, r) \) of \( P = f(N, r) \) about \( (N, r) = (48, 9) \).
f. Use the local linearization to estimate \( P(51, 9) \).
g. Find the slope of the contour that passes through \( (N, r) = (48, 9) \).
h. Estimate at which interest rate \( r_1 \) the monthly payment \( P(51, r_1) \) equals \( P(48, 9) \).
i. Estimate the value of the second partial derivative \( \frac{\partial^2 P}{\partial N^2} (48, 9) \).
j. Explain the practical significance of the fact that this derivative is positive.

4. Consider the change to polar coordinates \( x = r \cos \Theta, y = r \sin \Theta \) and \( r = \sqrt{x^2 + y^2}, \Theta = \tan^{-1} \left( \frac{y}{x} \right) \).

a. Calculate \( \frac{\partial x}{\partial r} \) and \( \frac{\partial x}{\partial \Theta} \). Is \( \frac{\partial x}{\partial r} \frac{\partial r}{\partial \Theta} = 1 \)?
b. Express the differential \( dx \) as a function of \( r, \Theta, dr \), and \( d\Theta \).

BONUS: Suppose that a radar station measures the distance of an incoming object as \( r = 10 \) km with a possible error of no more than \( |dr| \leq 0.5 \) km, and its direction at \( \Theta = \frac{\pi}{6} \) with a possible error of no more than \( |d\Theta| \leq 0.1 \). Calculate the x-coordinate of the measured position \((x, y)\) and use the differential \( dx \) to estimate the possible error.