1. Consider the function \( f(x, y) = x^3 - y^3 + 3y^2 - 3x - 7 \).

You may earn up to **10 BONUS points** if you work this problem for \( g(x, y) = 171x^3 + 84x^2y - 63xy^2 + 172y^3 - 5625x - 7500y \) instead of the \( f(x, y) \) above.

Note, this alternative has nice solutions, but almost requires a computer.

a. Find all critical points of \( f(x, y) \).

b. Classify the critical points using second derivatives.

2. Consider the function \( z = \sqrt{x^2 + y^2} \).

a. Describe the shape of the graph in words.

b. Sketch vertical cross-sections of the graph, parallel to the \( xz \) plane for \( y = -2, -1, 0, 1, 2 \).

You may want to overlay these on the same set of axes.

c. Sketch a contour diagram of \( z \) and mark the gradient vectors at about a dozen different points.

**BONUS:** Use Lagrange multipliers to find the minimum of \( z \) subject to the constraint \( 3x + 4y = 25 \).

3. The table on the right shows the monthly payment \( P = f(N, r) \) (in $) for a $10,000 loan with an annual interest rate of \( r \)% to be paid off over \( N \) months.

<table>
<thead>
<tr>
<th>( N )</th>
<th>8.50</th>
<th>8.75</th>
<th>9.00</th>
<th>9.25</th>
<th>9.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>205.17</td>
<td>206.37</td>
<td>207.58</td>
<td>208.80</td>
<td>210.02</td>
</tr>
<tr>
<td>54</td>
<td>223.50</td>
<td>224.70</td>
<td>225.89</td>
<td>227.10</td>
<td>228.30</td>
</tr>
<tr>
<td>48</td>
<td>246.48</td>
<td>247.67</td>
<td>248.85</td>
<td>250.04</td>
<td>251.23</td>
</tr>
<tr>
<td>42</td>
<td>276.10</td>
<td>277.27</td>
<td>278.45</td>
<td>279.62</td>
<td>280.80</td>
</tr>
<tr>
<td>36</td>
<td>315.68</td>
<td>316.84</td>
<td>318.00</td>
<td>319.16</td>
<td>320.33</td>
</tr>
</tbody>
</table>

a. Is \( f \) a linear function? Explain why or why not.

b. What are the units of \( \frac{\partial P}{\partial N}(48, 9) \) and \( \frac{\partial P}{\partial r}(48, 9) \)?

c. Estimate the values of \( \frac{\partial P}{\partial N}(48, 9) \) and \( \frac{\partial P}{\partial r}(48, 9) \).

d. Explain the practical meanings of \( \frac{\partial P}{\partial N}(48, 9) \) and \( \frac{\partial P}{\partial r}(48, 9) \), and the significance of their signs (+/−).

e. Find a formula for the local linearization \( L(N, r) \) of \( P = f(N, r) \) about \( (N, r) = (48, 9) \).

f. Use the local linearization to estimate \( P(51, 9) \).

g. Find the slope of the contour that passes through \( (N, r) = (48, 9) \).

h. Estimate at which interest rate \( r_1 \) the monthly payment \( P(51, r_1) \) equals \( P(48, 9) \).

i. Estimate the value of the second partial derivative \( \frac{\partial^2 P}{\partial N^2}(48, 9) \).

j. Explain the practical significance of the fact that this derivative is positive.

4. Consider the change to polar coordinates \( x = r \cos \Theta, y = r \sin \Theta \) and \( r = \sqrt{x^2 + y^2}, \Theta = \tan^{-1} \left( \frac{y}{x} \right) \).

a. Calculate \( \frac{\partial x}{\partial r} \) and \( \frac{\partial x}{\partial \Theta} \). Is \( \frac{\partial y}{\partial r} \frac{\partial r}{\partial x} = 1 \)?

b. Express the differential \( dx \) as a function of \( r, \Theta, dr, \) and \( d\Theta \).

**BONUS:** Suppose that a radar station measures the distance of an incoming object as \( r = 10 \text{km} \) with a possible error of no more than \( |dr| \leq 0.5 \text{km} \), and its direction at \( \Theta = \frac{\pi}{6} \) with a possible error of no more than \( |d\Theta| \leq 0.1 \). Calculate the \( x \)-coordinate of the measured position \( (x, y) \) and use the differential \( dx \) to estimate the possible error.