Show your work - explain what you are doing.  
It is YOUR responsibility to demonstrate that you have mastered the material of this class.

You may bring in one page of formulas – which you have to turn in with your solutions.

1.a. Use the method of Lagrange multipliers to find the minimum of \( f(x, y) = x^2 + y^2 \) subject to the constraint \( 4x + 3y = 50 \).

b. Sketch the graph of the constraint together with several level curves of \( f(x, y) \).
Label the diagram, and mark the point where the minimum occurs.

2. Work only one of the following two:

i.) Solve problem 1.a again, using only techniques from single-variable calculus. Explain the advantages of either approach. Explain why the single-variable calculus technique fails in the problem where the constraint has been replaced by \( 4x + 3y + \sin(x + y) = 50 \), yet the method of Lagrange multipliers still works.

ii.) Explain the reasoning that leads to the method of Lagrange multipliers. Specifically, suppose that at a point \((a, b)\) the constraint curve \( g(x, y) = 0 \) ((transversally)) crosses the level curves \( f(x, y) = \text{const} \) (and also \( \nabla f(a, b) \neq 0 \)). Explain why the minimum of \( f(x, y) \) subject to \( g(x, y) = 0 \) cannot possibly occur at \((a, b)\). Demonstrate how one arrives from here at the equation \( \nabla f(x, y) = \lambda \nabla g(x, y) \).

3.a. Given the data on the left \((x \text{ listed across, } y \text{ listed up and down})\) for values of a function \( f(x, y) \), estimate the definite integral \( \iint_{R} f \, dA \) where \( R \) is the rectangle with vertices \((0, 0), (20, 0), (0, 60), (20, 60)\).

b. Someone claims that \( \iint_{R} h \, dA \approx -13.45 \) where \( h(x, y) = e^{-x^2-y^2} \) and \( R \) is the disk \( R = \{(x, y): x^2 + y^2 \leq 1\} \). Explain why the estimate cannot possibly be close to the true value.

c. For bonus credit: Give very crude upper and lower bounds for the integral in 3.b. that require no work.

4. Let \( R \) be the region above the \( x \)-axis and inside the circle with center at the origin and radius \( r = 1 \).

a. Set up four iterated integrals (in rectangular and in polar coordinates, each in either order) for this region.
b. Explain in a few words (or with pictures), where the factor \( r \) in \( dA = r \, d\theta \, dr \) comes from.
c. Assuming constant mass-density \( \mu \) (mass per unit area) find the center of mass \((\bar{x}, \bar{y})\).

Hints: Without calculations explain why \( \bar{x} = 0 \). The formula for \( \bar{y} \) is \( \bar{y} = \frac{\iint_{R} x \mu \, dA}{\iint_{R} \mu \, dA} \).

5. Let \( R \) be the region above the cone \( z = \sqrt{3} \, r \) and inside the sphere with center at the origin and radius \( \rho = 2 \). Assuming constant mass density \( \mu \) find the moment of inertia \( I_z = \iiint_{R} \rho r^2 \, dV \) of \( R \) about the \( z \)-axis.

For bonus credit: Rewrite the integral of the last problem in several different ways (i.e., in different coordinates, and in different orders of integration). Compare where the resulting integrations get hard.