1.) A Riemann sum with $\Delta x = 0.2$ and $\Delta y = 0.5$, using the midpoints of the rectangles gives the approximation $\int_0^1 \int_0^1 f(x, y) dA \approx (1.2 + 1.9 + \ldots + 5) \cdot 0.2 \cdot 0.5 = 2.860$. Without any further information no error bounds are possible – the function could have arbitrarily large variation between the data points.

2.) The only critical point of $f$ lies at $(0, 0)$ which is outside the triangle. Consequently the minimum and maximum on the triangle must occur on its edges (or corners).
Inside the first quadrant $f_x = 2x$ and $f_y = 8y$ are positive, and hence $f$ is increasing along the upper and the right side from $C$ to $B$ and from $A$ to $B$.
The contour diagram suggests that the minimum occurs along the side $CA$. Formulate as a constrained optimization problem and use Lagrange multipliers: Write the constraint as $g(x, y) = 5 - x - y = 0$ and form the Lagrangian $F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = (x^2 + 4y^2) + \lambda(5 - x - y)$. Solve the system of equations

\[
\begin{align*}
0 &= F_x(x, y, \lambda) = 2x - \lambda \\
0 &= F_y(x, y, \lambda) = 8y - \lambda \\
0 &= F_\lambda(x, y, \lambda) = 5 - x - y
\end{align*}
\]

This is a linear system with unique solution $Q(x, y) = (4, 1)$. Since $f(Q) = 20 < f(A) = 25$ this is the minimum of $f$ along the entire line $y = 5 - x$, and consequently is the minimum on the triangle. The maximum then must occur at the corner $B$, namely $f(B) = 125$.

b.) In this case the global unconstrained minimum of $f$ at $(0, 0)$ lies inside the triangle, and is also the minimum on the new triangle. The maximum remains the same, as suggested by the contours (strictly speaking, this required some work to verify).

c.) The integral lies between the min and the max, each multiplied by the area, i.e between $20 \times 5^2/2 = 250$ and $125 \times 5^2/2 = 1562.5$.

d.) $\int_T f(x, y) dA = \int_0^5 \int_{5-x}^{5-y} (x^2 + 4y^2) dy dx = 781.25$. To get the average divide by the area: $f_{av} = 62.5$.

3.) The equation of the plane containing the points $BCD$ is $6x + 3y + z = 12$. Thus the iterated integrals are:

\[
\int_{R} f dV = \int_0^4 \int_0^{2-\frac{x}{3}} \int_0^{12-6x-3y} f(x, y, z) dz dy dx = \int_0^4 \int_0^{2-\frac{x}{3}} f(x, y, z) dz dy dx
\]

b.) In the case that $f(x, y, z) = x^2 + y^2$ the integral evaluates to $32$.

4.) The integral may be rewritten as iterated integrals as follows:

\[
f_{R} f dA = \int_0^3 \int_0^3 f(r \cos \Theta, r \sin \Theta) r dr d\Theta = \int_0^3 \int_0^{\pi} f(r \cos \Theta, r \sin \Theta) r d\Theta dr
\]

\[
= \int_0^3 \int_{\sqrt{9-x^2}}^{\sqrt{9-y^2}} f(x, y) dy dx + \int_0^3 \int_{\sqrt{9-y^2}}^{\sqrt{9-x^2}} f(x, y) dy dx
\]

\[
= \int_0^3 \int_{\sqrt{9-y^2}}^{\sqrt{9-x^2}} f(x, y) dy dx + \int_0^3 \int_{\sqrt{9-x^2}}^{\sqrt{9-y^2}} f(x, y) dy dx
\]

b.) Evaluate $\int_{R} x dA = \int_0^3 \int_0^3 r^2 \cos \Theta dr d\Theta = \frac{19}{6} \sqrt{2}$ and $\int_{R} y dA = \frac{1}{4} \pi (3^2 - 2^2) = \frac{5}{2} \pi$ (one quarter of the difference of the areas of two circles). The quotient of these integrals yields $\bar{x} = \bar{y} \approx 2.28$, which is close to the inner edge, but still well within the quarter annulus.