Test 3  Calculus and Analytic Geometry 3  11/10/1993

Sample solutions

1.) The gradients are perpendicular to the level curves. If they are not parallel, then the level curve \( g(x, y) = 0 \) will cross the curve \( f(x, y) = c = f(a, b) \) "transversally", and thus will cross level curves \( f(x, y) = k \) both with \( k \) larger and with \( k \) smaller than \( c = f(a, b) \) (assuming that \( \nabla f(a, b) \neq \emptyset \)). Therefore \( f(x, y) \) can have neither a maximum nor a minimum at \((a, b)\) (with the constraint \( g(x, y) = 0 \)).

2.a) After completing squares \( f(x, y) = (x - 4)^2 + (y - 2)^2 - 1 \). The graph is a circular paraboloid opening upwards with vertex at \((4, 2, -1)\). b) The level curves are concentric circles centered at \( P = (4, 2) \), see the sketch below.

c.) From the formula in a.) it is clear that the global minimum occurs at \( P = (4, 2) \), and that there are no other critical points. Alternatively, \( f_x(x, y) = 2x - 8 \) and \( f_y(x, y) = 2y - 4 \). Solving \( f_x(x, y) = f_y(x, y) = 0 \) yields \((x, y) = (4, 2)\). The second derivatives \( f_{xx}(x, y) = 2 \), \( f_{yy}(x, y) = 2 \), \( f_{xy}(x, y) = 0 \) are constant. \( D(x, y) = 4 \) is positive and therefore there is either a local max or a local min at any critical point. Since \( f_{xx}(4, 2) > 0 \) it is a local minimum. This is also the global minimum, and there is no local or global maximum. d.) In this case \( f_{xy} = 87 \) is much larger than either \( f_{xx}(x, y) \) or \( f_{yy}(x, y) \). Therefore \( D(x, y) \) is clearly negative, and the graph must be a saddle (there is again exactly one critical point).

3.a.) The area of a triangle is \( \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 3 \times 2 = 3 \). (For triangles in more general position (tilted etc.) use cross-products: area= \( \frac{1}{2} |\mathbf{AB} \times \mathbf{AC}| \), considered as vectors in three dimensional space.) b.) From the contours it appears that the maximum occurs at the corner \( B(1, 1) \) that is furthest from \( P \) and that the minimum occurs along the edge \( AC \). The minimum and maximum cannot occur in the interior of the triangle since there are no critical points of \( f(x, y) \) here.

Evaluating \( f(x, y) \) at the corners yields \( f(A) = 4 \), \( f(B) = 9 \) and \( f(C) = 0 \). Along the edge \( BC \) \( y = 1 \) is constant, and if \( k(x) = f(x, 1) = (x - 4)^2 \), then \( k''(x) = 2(x - 4) < 0 \) shows that \( k \) and hence \( f \) is decreasing from \( B \) to \( C \). Along the edge \( AB \) \( y = 2x - 1 \) and thus \( f(x, 2x - 1) = h(x) = (x - 4)^2 + 2(2x - 1)^2 - 1 \) has derivative \( h''(x) = 2(4x - 3) = 10x - 20 = 10(x - 2) \) is again decreasing from \( B \) to \( A \). Finally, along the edge \( AC \), we use Lagrange multipliers, just to show how to do it: The equation of the line through \( A \) and \( C \) is \( y = 5 - x \), and we consider the constraint \( g(x, y) = y + x - 5 = 0 \). Solve the system of 3 equations \( \nabla f(x, y) + \lambda \nabla g(x, y) = 0 \) together with \( g(x, y) = 0 \) for the 3 unknowns \( x, y, \lambda \):

\[
\begin{align*}
 f_x(x, y) + \lambda g_x(x, y) &= 2x - 8 + 1 \cdot \lambda &= 0 \\
 f_y(x, y) + \lambda g_y(x, y) &= 2y - 4 + 1 \cdot \lambda &= 0 \\
 g(x, y) &= x + y - 5 &= 0 \\
\end{align*}
\]

This linear system has a unique solution \((x, y, \lambda) = (3.5, 1.5, 1)\). The point \((3.5, 1.5)\) is clearly the point closest to \( P(4, 2) \) along the line \( AC \) (the line \( PQ \) is perpendicular to \( AC \)) and as \( f(x, y) \) is essentially distance of \((x, y)\) from \( P \) squared, this is the desired minimum. The minimum value is \( f(Q) = -\frac{1}{2} \). d.) The parallelogram now contains the global minimum of \( f(P) = -1 \) and this is also the constrained minimum.

4.a.) The integral must be between \(-\frac{1}{2} \cdot 3 = -\frac{3}{2} \) and \( 9 \cdot 3 = 27 \), the \( \min / \max \) of \( f(x, y) \) on \( T \) multiplied by its area.

b.) As one side of \( T \) is aligned with the \( x \)-axis it is easiest to first integrate with respect to \( x \):

\[
\int_1^3 \int_{x+1}^{5-x} f(x, y) \, dy \, dx
\]

c.) Using the following commands in MAPLE: \( z := (x-4) \wedge 2 + (y-2) \wedge 2 - 1; \ int\left(int(z, x=(y+1)/2..5-y), y=1..3\right) \); to obtain the value \( \frac{15}{8} \) for the integral, which yields the average value \( \frac{15}{8} \cdot \frac{1}{2} = 2.5 \) of \( f(x, y) \) over the triangle. This appears quite reasonable (it must be between \(-\frac{1}{2} \) and \( 9 \), and the concavity and shape of the triangle suggest that it is closer to the lower value).

5.a.) The equations of the planes of the sides are: bottom: \( z = 0 \), back \( x = 0 \), left \( y = 0 \), and top=front: \( 4x + 2y + z = 12 \).

b.) Considering horizontal slices \( z = \text{const.} \) gives \( z \) between 0 and 12 (bottom and top slice). On each slice \( x \) goes from the back \( x = 0 \) to the front \( x = (12 - 2y - z)/4 \), and \( y \) goes from \( y = 0 \) to the corner that is the intersection of the front \( 4x + 2y + z = 12 \) and the back \( x = 0 \). Combining these and solving for \( y = (12 - z)/2 \).

For a change, in the second case project on the \( xy \)-plane \( z = 0 \). The shadow is a triangle with sides \( x = 0, y = 0, \) and \( y = 6 - 2x \). Integrating last with respect to \( x \) gives \( f(x, y) \) from \( 0 \) to 3, and for each fixed \( x, y \) goes from \( y = 0 \) to \( y = 6 - 2x \). For each fixed pair \((x, y)\) in the shadow \( z \) goes from the bottom \( z = 0 \) to the top \( z = 12 - 4x - 2y; \)

\[
\int_0^{12} \int_0^{6-x/2} \int_{3-y/2-4/4}^{3-y/2} f(x, y, z) \, dx \, dy \, dz = \int_0^{6-x} \int_0^{12-4x} f(x, y, z) \, dy \, dx.
\]

6.) This problem may reappear on the next test or the final exam. You may also turn in solutions as a special project, provided they are neatly written up, and you are willing to answer questions about them in my office.