1.a.) Let \( \vec{v}_3 = 2\vec{i} \) and \( \vec{v}_{12} = 2\vec{j} \). Then \( \vec{v}_1 = (2 \cos \frac{\pi}{3})\vec{i} + (2 \sin \frac{\pi}{3})\vec{j} = \vec{i} + \sqrt{3}\vec{j} \), \( \vec{v}_2 = 3\vec{i} + \vec{j} \) etc.
Define \( \vec{u} = (\vec{v}_3 + (\vec{i}_1 + \vec{v}_7)) + (\vec{v}_3 + (\vec{i}_2 + \vec{v}_4)) = (2 + 2\sqrt{3})(\vec{i} + \vec{j}) \) which has length \( |\vec{u}| = 2\sqrt{2}(1 + \sqrt{3}) \), i.e. \( \vec{u} \) is approximately 3.864 times as long as any single vector \( \vec{v}_n \).
We claim that this is the longest possible. If we add a vector \( \vec{u} \) to a vector \( \vec{w} \) then \( \vec{u} + \vec{w} \) is longer than \( \vec{w} \) only if \( |\vec{w} + \vec{u}|^2 = (\vec{w} + \vec{u}) \cdot (\vec{w} + \vec{u}) = |\vec{w}|^2 + (2\vec{w} + \vec{u}) \cdot \vec{u} \) is larger than \( |\vec{w}|^2 \), i.e. the angle between \( \vec{u} \) and \( 2\vec{w} + \vec{u} \) is less than \( \frac{\pi}{2} \). For example take \( \vec{u} = \vec{v}_5 = \vec{i} - \sqrt{3}\vec{j} \), then \( 2\vec{w} + \vec{v}_5 = (5 + 4\sqrt{3})\vec{i} + (4 + 3\sqrt{3})\vec{j} \) and \( \vec{v}_5 \cdot (2\vec{w} + \vec{v}_5) = (5 + 4\sqrt{3}) + (-4\sqrt{3} - 9) = -4 < 0 \), i.e. \( \vec{u} \) is shorter than \( \vec{v} \).
(A complete argument has to include all other possibilities, too much for here.)

b.) With \( \vec{v} \) as above, \( \vec{u}_1 = \vec{u} + v_{12} \). By symmetry \( \vec{u}_2 + \vec{u}_4 + \vec{u}_6 + \ldots + \vec{u}_{12} = 6\vec{u}_{12} = 3\vec{u}_{12} \).

2.a.) Since always \( \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \alpha \) (where \( \alpha \) is the angle between \( \vec{v} \) and \( \vec{w} \)), if \( \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \) then \( \cos \alpha = 1 \). This means that \( \alpha = 0 \), or \( \vec{v} \) and \( \vec{w} \) point in the same direction.

b.) Two sides of the triangle are \( \overline{BA} = -\vec{i} - \vec{j} \) and \( \overline{BC} = \vec{j} + \vec{k} \), hence \( \overline{N} = \overline{BA} \times \overline{BC} = -\vec{i} + \vec{j} - \vec{k} \) is perpendicular to the triangle, and the area of the triangle is \( \frac{1}{2} |\overline{N}| = \frac{1}{2} \sqrt{3} \approx 0.8660 \).

c.) Since \( \overline{AD} \cdot \overline{N} = -3 + 8 = 5 \neq 0 \) the point \( D \) does not lie in the plane.
(Alternatively, the long way: The equation of the plane thru \( A, B, C \) is e.g. \( \overline{N} \cdot \overline{R} = \overline{N} \cdot \overline{OA} \). In the long form this equation reads \( x - y + z = 0 \) after multiplying by \( (\overline{1}, -\overline{1}, \overline{1}) \). The coordinates of \( D \) clearly do not satisfy the equation.)

d.) Name the corners of the triangle \( A, B, C \) with \( C \) opposite to the longest side. Let \( \vec{u} = \overline{CA} \) and \( \vec{v} = \overline{CB} \). Then \( \vec{u} - \vec{u} = \overline{A}\overline{B} \), and \( |\vec{u}| = b, |\vec{v}| = a \) and \( |\vec{u} - \vec{u}| = c \). Taking the dot-product we get \( c^2 = (\vec{u} - \vec{u}) \cdot (\vec{u} - \vec{u}) = \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} = a^2 + b^2 \), using that \( \vec{u} \cdot \vec{u} = 0 \) since \( \vec{u} \) and \( \vec{v} \) are perpendicular.

3.a.) If \( f(x, y) = x^2 y^2 \) then \( \vec{\nabla} f(1,1) = 2\vec{i} + 3\vec{j} \) and the direction of the steepest increase is \( \vec{u} = \frac{1}{\sqrt{13}}(2\vec{i} + 3\vec{j}) \).

b.) \( f(1+\Delta x, 1+\Delta y) \approx f(1,1) + \vec{\nabla} f(1,1) \cdot \vec{u} \) where \( \vec{u} = 0.065\vec{i} - 0.037\vec{j} \). Consequently, \( f(1.05, 0.97) \approx 1 + 0.1 - 0.09 = 1.01 \).

4.) The contour lines in the first two cases are circles centered at the origin. In the first case they are equally spaced (the graph is a cone opening upwards), and in the second case they are increasingly closer spaced as one gets further away from the origin (the graph may be the upper half sphere or a paraboloid opening downward).

The third gradient field corresponds to a plane with parallel, equally spaced straight lines as contours. These have a slope of approximately \(-\frac{2}{3}\).

5.a.) Since \( D_{x}f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{i} \) we have \( 0.8f_x(a,b) + 0.6f_y(a,b) = 2 \), and similarly \( 0.6f_x(a,b) + 0.8f_y(a,b) = 3 \).
These are two linear equations in the two unknown \( f_x(a,b) \) and \( f_y(a,b) \).
E.g. multiply both sides of the first equation by \( 0.8 \) and the second by \( -0.6 \), and add to get \( 0.64 - 0.36)f_x(a,b) = 1.6 - 1.2, i.e. \( f_x(a,b) = \frac{5}{12} \approx 0.7142 \) and similarly, \( 0.64 - 0.36)f_y(a,b) = 2.4 - 1.2, i.e. \( f_y(a,b) = \frac{5}{12} \approx 4.286 \).

Bonus problem: If the partial derivatives are continuous at \((a,b)\), yes, then all tangent lines to the graph at this point lie in one plane, which we call the tangent plane.

One may argue as follows: The tangent lines in the \( x \)- and the \( y \)-direction have slopes \( f_x(a,b) \) and \( f_y(a,b) \), respectively, and hence the vectors \( \vec{v} = \vec{i} + f_x(a,b)\vec{k} \) and \( \vec{w} = \vec{j} + f_y(a,b)\vec{k} \) point in the direction of these two tangent lines. These two lines span a plane with normal vector \( \overline{N} = \vec{v} \times \vec{w} = f_x(a,b)\vec{k} - f_y(a,b)\vec{j} + f_y(a,b)\vec{i} \).

Now consider the tangent line at \((a,b)\) in an arbitrary direction \( \vec{z} = z_1\vec{i} + z_2\vec{j} \). By definition its slope is \( D_xf(a,b) \), and hence the vector \( \vec{u} = \vec{z} + D_xf(a,b)\vec{k} \) points in the direction of the corresponding tangent line. It will lie in the same plane as \( \vec{v} \) and \( \vec{w} \) if and only if \( \vec{u} \perp \overline{N} \). Compute the dot-product \( \vec{u} \cdot \overline{N} = (\vec{z} + D_xf(a,b)\vec{k}) \cdot (-f_x(a,b)\vec{k} - f_y(a,b)\vec{j} + f_y(a,b)\vec{i}) = -z_1f_x(a,b) - z_2f_y(a,b) + D_xf(a,b) \). But this is zero since \( D_xf(a,b) = \vec{\nabla} f(a,b) \cdot \vec{i} \).