3.14 Tensors and alternating forms

The goal of this section is to prepare for an alternative way of characterizing integrability, which relies on differential forms (as opposed on tangent vector fields). The algebraic structure of differential $k$-forms, i.e. their products and exterior derivatives require some preliminary constructions involving tensors. This section can only give a very brief survey of some basic concepts, terminology and select fundamental properties. The language of tensors developed here will also be beneficial later for precise descriptions of objects in Riemannian geometry.

To motivate tensors (and subsequently tensor fields) we explore two basic examples – linear maps between two vector spaces and quadratic forms on a vector space. The key points will be to contrast multi-linear with linear maps, and contrast Cartesian products and direct sums with the spaces of linear maps and tensor products. The examples motivate an algebraic characterization of tensor products as objects that naturally allow one to factor all multi-linear (bilinear) maps between two vector spaces and quadratic forms on a vector space. The key points will be to contrast multi-linear with linear maps, and contrast Cartesian products and direct sums with the examples motivate an algebraic characterization of tensor products as objects that naturally allow one to factor all multi-linear (bilinear) maps. Here we restrict the attention to finite dimensional vector spaces over the field $k = \mathbb{R}$ of real numbers (similar notions may be developed for infinite dimensions and for modules over commutative rings). Note that there are many settings in differential geometry where a vector space like the space $\Gamma^\infty(M)$ of smooth vector fields may not only be considered as a vector space over the field $\mathbb{R}$, but also its structure as a module over the ring $\mathcal{C}^\infty(M)$ is utilized often, for example one may consider $\Gamma^\infty(M)$ and $\Omega^1(M)$ not only as vector spaces over $\mathbb{R}$, but it also has the structure of a module over the ring $\mathcal{C}^\infty(M)$ of smooth functions – and consequently one needs to meticulously distinguish different tensor products such as

$$\Gamma^\infty(M) \otimes_\mathbb{R} \Omega^1(M) \quad \text{and} \quad \Gamma^\infty(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^1(M)$$

However, in this section we will mostly consider tensor products at individual points $p \in M$, which only involve vector spaces like $T_p M$ and $T_p^* M$ over the field $\mathbb{R}$.

Let $V$ and $W$ be finite dimensional vector spaces. Write $\text{Hom}(V, W)$ for the vector space of linear maps from $V$ to $W$. To exhibit the similarity between the Cartesian product $V \times W$ and the space of linear maps $\text{Hom}(V, W)$, define a bilinear map $\Phi : V^* \times W \rightarrow \text{Hom}(V, W)$. To characterize $\Phi(\lambda, w) \in \text{Hom}(V, W)$ (for every $\lambda \in V^*$ and every $w \in W$) we need to specify $\Phi(\lambda, w)(v) \in W$ for every $v \in V$. The (only, nontrivial, natural) way to do this, is to define

$$\Phi : V^* \times W \rightarrow \text{Hom}(V, W) \quad \text{for} \quad \lambda \in V^*, \ v \in V \quad \text{and} \quad w \in W \quad \text{by} \quad \Phi(\lambda, w)(v) = \lambda(v) w$$

Bilinearity means that for any $\lambda_1, \lambda_2 \in V^*$, $w_1, w_2 \in W$ and and $c \in k = \mathbb{R}$,

$$\Phi(c \lambda_1 + \lambda_2, w) = c \cdot \Phi(\lambda_1, w) + \Phi(\lambda_2, w)$$
$$\Phi(\lambda, cw_1 + w_2) = c \cdot \Phi(\lambda, w_1) + \Phi(\lambda, w_2)$$

(121)

Note that the map $\Phi$ is not one-to-one – indeed for any $c \in \mathbb{R} \setminus \{0\}$, $\lambda \in V^*$, and $w \in W$, the linear maps $\Phi(\lambda, w) = \Phi(c \lambda, \frac{1}{c} w)$ are identical. Moreover, the additive structure of the direct sum $V^* \oplus W$ (which as a set is the same because $V^* \times W$) i.e. $(\lambda_1, w_1) + (\lambda_2, w_2) = (\lambda_1 + \lambda_2, w_1 + w_2)$ is also mismatched by the bilinear (as opposed to linear) map $\Phi$ as

$$\Phi(\lambda_1 + \lambda_2, w_1 + w_2)(v) = (\lambda_1 + \lambda_2)(v) \cdot (w_1 + w_2)$$
$$= \lambda_1(v) w_1 + \lambda_1(v) w_2 + \lambda_2(v) w_1 + \lambda_2(v) w_2$$
$$\neq \lambda_1(v) w_1 + \lambda_2(v) w_2$$

$$= \Phi(\lambda_1, w_1)(v) + \Phi(\lambda_2, w_2)(v)$$

(122)
**Exercise 3.48** Suppose that \( \{\lambda^1, \ldots, \lambda^m\} \) and \( \{w_1, \ldots, w_n\} \) are bases for vector spaces \( V^* \) and \( W \), respectively. With \( \Phi \) as defined above, show that \( \{\Phi(\lambda^i, w_j) : i \leq m, j \leq n\} \) is a basis for \( \text{Hom}(V, W) \).

As a consequence of the exercise we conclude that for every vector space \( Z \) every bilinear map \( \beta : V^* \times W \to Z \) factors uniquely into a composition \( \beta = \Psi_\beta \circ \Phi \) with \( \Phi \) as above and \( \Psi_\beta \) a uniquely determined linear map \( \Psi_\beta : \text{Hom}(V, W) \to Z \).

As a second example consider quadratic (bilinear) forms \( Q : V \times V \to k = \mathbb{R} \) on a finite dimensional vector space \( V \). Bilinearity means that for \( v, v_1, v_2 \in V \) and \( c \in \mathbb{R} \)

\[
Q(cv_1 + v_2, v) = cQ(v_1, v) + Q(v_2, v) \\
Q(v, cv_1 + c_2) = cQ(v, v_1) + Q(v, v_2) \tag{123}
\]

We will factor \( Q \) as a linear map composed with a universal bilinear map . . .

**Exercise 3.49** Suppose \( V, Q, \) and \( \{\lambda^1, \ldots, \lambda^n\} \) are as above. Show that there exist uniquely determined \( a_{ij} \in \mathbb{R} \) such that

\[
Q(v, v') = \sum_{i,j=1}^n a_{ij} \lambda^i(v) \cdot \lambda^j(v') \quad \text{for all } v, v' \in V. \tag{124}
\]

In the following we shall characterize the space in which \( \sum_{i,j=1}^n a_{ij} \lambda^i(\cdot) \lambda^j(\cdot) \) naturally lives in. Note, if \( c \neq 0, 1 \) then \((\lambda, \lambda)\) and \((c\lambda, 1/2 \lambda)\) are different as elements of \( V^* \times V^* \). However, they determine the same quadratic form \( Q \) via the map \( \Phi \) from \( V^* \times V^* \) to the vector space of quadratic bilinear forms on \( V \), defined by

\[
\Phi(\lambda, \lambda')(v, w) = \lambda(v) \cdot \lambda'(w) \tag{125}
\]

Both examples suggest to build from the Cartesian product of vector spaces a new vector space with a linear structure analogous to that of spaces of linear maps. More specifically we want to identify (on \( V \times W \)) e.g. \((cv, w)\) with \((v, cw)\) and have an additive structure such that \((v + v', w + w') = (v, w) + (v', w) + (v', w') \).
Here multi-linearity means that for \( v_1 \in V_1, \ldots, v_i, v'_i \in V_i, \ldots, v_k \in V_k \) and \( c \in \mathbb{R} \)
\[
\mu(v_1, \ldots, v_{i-1}, cv_i + v'_i, v_{i+1} \ldots v_k) = c \cdot \mu(v_1, \ldots, v_i, \ldots, v_k) + \mu(v_1, \ldots, v_i', \ldots, v_k) \quad (126)
\]
Using standard algebraic arguments one easily shows that such a universal space \( T \) and associated map \( \Phi \) always exist, and they are unique, up to isomorphisms \([\text{Following Sternberg}]\) the construction starts with the Cartesian product \( V = V_1 \times \ldots \times V_k \), then considers the free vector space \( \overline{V} \) generated by \( V \) (i.e. the set of all formal linear combinations of elements in \( V \)). Let \( \overline{R} \subseteq \overline{V} \) be the subspace generated by all elements of the from
\[
c(v_1, \ldots, v_k) - (v_1, \ldots, v_{i-1}, cv_i, v_{i+1} \ldots v_k) \quad \text{and all elements of the form}
\]
\[
(v_1, \ldots, v_{i-1}, v_i + v'_i, v_{i+1} \ldots v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v_i', \ldots, v_k) \quad (127)
\]
where \( c \in \mathbb{R}, v_j \in V_j \) for \( 1 \leq j \leq k \) and \( v'_i \in V_i \) for some \( i \leq k \).

**Definition 3.24** For \( V_1, \ldots, V_k, \overline{V} \) and \( \overline{R} \) as above define the tensor product of the spaces \( V_j \) as the quotient
\[
V_1 \otimes \ldots \otimes V_k = \overline{V} / \overline{R}. \quad (128)
\]
Define the multi-linear map \( \Phi: V_1 \times \ldots \times V_k \mapsto V_1 \otimes \ldots \otimes V_k \) which maps \( (v_1, \ldots, v_k) \) to its coset mod \( \overline{R} \) (viewing \( (v_1, \ldots, v_k) \) as an element of \( \overline{V} \)). Write \( v_1 \otimes \ldots v_k \) for \( \Phi(v_1, \ldots, v_k) \).

For finite dimensional vector spaces \( V_1, \ldots, V_k \) over \( \mathbb{R} \) the tensor product \( T = V_1 \otimes \ldots \otimes V_k \) (together with the canonical map \( \Phi \)) is the desired universal space:

**Proposition 3.32** For every vector space \( W \) and every multi-linear map \( \mu : V_1 \times \ldots \times V_k \mapsto W \) there exists a unique linear map \( \Psi_\mu : V_1 \otimes \ldots \otimes V_k \mapsto W \) such that \( \mu = \Psi_\mu \circ \Phi \) (with \( \Phi \) as above).

**Proof.** Suppose \( \mu : V_1 \times \ldots \times V_k \mapsto W \) is a multi-linear map. Define a map \( \overline{\Psi}_\mu : \overline{V} \mapsto W \) on generators by \( \overline{\Psi}_\mu(v_1, \ldots, v_k) = \mu(v_1, \ldots, v_k) \) and extend linearly. Since \( \mu \) is multi-linear, the restriction of \( \overline{\Psi}_\mu \) to \( \overline{R} \) is identically zero, and hence this defines a map \( \Psi_\mu \) on the quotient. Uniqueness is clear as the image of \( \Phi \) contains a set of generators for \( V_1 \otimes \ldots \otimes V_k \) and any other such map \( \Psi_\mu \) necessarily must agree with \( \Psi_\mu \) on this set, hence agree with \( \Psi_\mu \) everywhere. \( \blacksquare \)

Note that, as defined above, the tensor product is naturally associative, it is not commutative as clearly \( v \otimes w \) and \( w \otimes v \) denote different objects in different spaces – but there is a natural isomorphism between e.g. \( V \otimes W \) and \( W \otimes V \).

It is instructive to connect this notion of tensor products to familiar objects like matrices. Going back to the first example, suppose \( \beta = \{v_1, \ldots, v_m\} \), and \( \gamma = \{w_1, \ldots, w_n\} \) are bases for vector spaces \( V \) and \( W \) and \( \beta^* = \{\lambda^1, \ldots, \lambda^m\} \) is the associated dual basis for \( V^* \). Moreover, suppose \( L \in \text{Hom}(V, W) \) is a linear map from \( V \) to \( W \). Then there exist unique \( a_{ij}^* \in \mathbb{R} \), such that for all \( v \in V \), \( L(v) = \sum_{j=1}^n a_{ij}^* \lambda^j(v) w_i \). Pictorially identify the product \( \lambda^j \otimes w_i \) with the matrix

\[
\lambda^j \otimes w_i \quad \longleftrightarrow \quad \begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0
\end{pmatrix} \quad \leftarrow \text{i-th row} \quad \leftarrow \text{j-th column} \quad (129)
\]
Consider the example of \( i \cdot \lambda^j \) written as a row vector and \( w_i \) written as a column vector then their tensor product \( \lambda^j \otimes w_i \) corresponds to the matrix product "\( w_i \cdot \lambda^j \)" in reversed order [...]. The linear combination of such matrices with coefficients \( a^i_j \) yields a matrix \( A \) which is the matrix representation of the linear map \( L \) with respect to the bases \( \beta \) and \( \gamma \). In the case that \( \beta \) and \( \gamma \) are the standard bases for \( V = \mathbb{R}^m \) and \( W = \mathbb{R}^n \) one usually writes \( v_i = w_i = e_i \) and \( \lambda^j = e^j \) and the matrix displayed above corresponds to \( e^j \otimes e_i \). [[As above, some may prefer to change the order of the factors and write \( e_i \otimes e^j \) to have a closer match to matrix-products]]

**Exercise 3.50** Consider the example of \( V = W = \mathbb{R}^2 \) with standard basis \( \{e_1, e_2\} \), and the bilinear map \( \Phi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \otimes \mathbb{R}^2 \) defined on generators by \( \Phi(e_i, e_j) = e_i \otimes e_j \). Identify \( e_i \otimes e_j \) with the matrix whose entry in the \( i \)-th row and \( j \)-th column is 1 and whose other entries are all zero. 

(i) Show that there does not exist \((v, w) \in V \times W \) such that \( \Phi(v, w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

(ii) Describe the set of all matrices that lie in the image of \( \Phi \). [[Note that not every matrix is in the image, but every \( 2 \times 2 \) matrix is a linear combination of \( \Phi(e_i, e_j) \) with \( i, j = 1, 2 \) (all combinations).]]

(iii) Calculate the derivative of \( \Phi \), and its rank.

In the classical literature one often finds a characterization of tensors as objects that obey certain rules for transformations. In the context of the preceding discussion, these refer to changes of bases in the factors \( V_i \). In the general context of differential manifolds these changes of bases in the tangent and cotangent bases arise from changes of local coordinates on the manifold. For illustration of the kinds of expression involved consider a single vector space \( V \) with bases \( \beta = \{v_1, \ldots, v_n\} \) and \( \gamma = \{w_1, \ldots, w_n\} \). Let \( V^* \) be the dual space with associated dual bases \( \beta^* = \{\lambda^1, \ldots, \lambda^n\} \) and \( \gamma^* = \{\mu^1, \ldots, \mu^n\} \). For illustration consider the tensor product \( T = V \otimes V \otimes V^* \). Bases for this space is given by \( \{v_i \otimes v_j \otimes \lambda^k : i, j, k \leq n\} \) and \( \{w_i \otimes w_j \otimes \mu^k : i, j, k \leq n\} \). Thus any element \( z \in T \) can be written uniquely as linear combination

\[
z = \sum_{i,j,k} a^{ij}_{\ k} \ v_i \otimes v_j \otimes \lambda^k = \sum_{i,j,k} b^{ij}_{\ k} \ w_i \otimes w_j \otimes \mu^k \quad (130)
\]

The transformation rules refer to the identities that relate the coordinates \( a^{ij}_{\ k} \) to the coordinates \( b^{ij}_{\ k} \). Specifically, there exist \( c^j_i \), \( d^i_j \in \mathbb{R} \) such that

\[
w_i = \sum_j c^j_i \ v_j \quad \text{and} \quad \mu^i = \sum_j d^i_j \ \lambda^j \quad \text{and} \quad (131)
\]

The duality of the bases means that \( \lambda^i(v_j) = \mu^i(w_j) = \delta_{i,j} \). The following small calculation reaffirms that the matrices \( c^j_i \) and \( d^i_j \) are inverses of each other:

\[
\delta_{i,j} = \mu^i(w_j) = (\sum_k d^i_k \lambda^k)(\sum_\ell c^j_\ell \ v_\ell) = \sum_k d^i_k c^j_\ell \cdot \lambda^k(v_\ell) = \sum_k d^i_k c^j_\ell \cdot \lambda^k(v_\ell) = \sum_k d^i_k c^j_\ell \quad (132)
\]

To obtain the transformation rules from \( a^{ij}_{\ k} \) to \( b^{ij}_{\ t} \) calculate

\[
z = \sum_{i,i,k} a^{ij}_{\ k} \ v_i \otimes v_j \otimes \lambda^k = \sum_{r,s,t} \left( \sum_{i,i,k} a^{ij}_{\ k} \cdot c^r_i \cdot c^s_j \cdot d^t_k \right) \ w_r \otimes w_s \otimes \mu^t \quad (133)
\]
In addition to the standard identification of $\text{Hom}(V,W)$ with $V^* \otimes W$ there are many similar, useful relations.

- A natural isomorphism between $\text{Hom}(U \otimes V, W)$ and $\text{Hom}(U, \text{Hom}(V, W))$ is provided by

$$\Phi(F)(u)(v) = F(u \otimes v)$$  \hspace{1cm} (134)

- A natural isomorphism between $\text{Hom}(U, V) \otimes W$ and $\text{Hom}(U, V \otimes W)$ is provided by

$$\Phi(F \otimes w)(u) = F(u) \otimes w$$  \hspace{1cm} (135)

In the special case that $V = \mathbb{R}$ this yields the isomorphism between $U^* \otimes W$ and $\text{Hom}(U, W)$.

- A natural isomorphism between $U \otimes \text{Hom}(V, W)$ and $\text{Hom}(\text{Hom}(U, V), W)$ is provided by

$$\Phi(u \otimes G)(F) = F(G(u))$$  \hspace{1cm} (136)

The special case that $V = W = \mathbb{R}$ yields the isomorphism between $U^{**}$ and $U$.

- A natural isomorphism between $\text{Hom}(U, V) \otimes \text{Hom}(W, Z)$ and $\text{Hom}(U \otimes V, W \otimes W)$ is provided by

$$\Phi(F \otimes G)(u \otimes v)) = F(u) \otimes G(v)$$  \hspace{1cm} (137)

The special case that $V = Z = \mathbb{R}$ yields the isomorphism between $U^* \otimes V^*$ and $(U \otimes V)^*$.

Our primary interest is in tensor products of tangent spaces and co-tangent spaces, i.e. in tensor products where each factor is either the same vector space $V$ or its dual $V^*$. Due to the lack of commutativity there are may different higher order products that all arise from a single vector space and its dual. Taking formal linear combinations one conveniently may combine all possible tensor products into an algebra of tensor products:

**Definition 3.25** [[Sternberg defines]] the tensor algebra [[?]] of a vector space $V$ is the direct sum of all tensor products of $V$ and $V^*$, i.e.

$$\mathcal{T}(V) = \mathbb{R} \oplus V \oplus V^* \oplus V \otimes V \oplus V^* \otimes V \oplus V^* \otimes V \oplus V^* \otimes V^* \oplus \ldots$$  \hspace{1cm} (138)

We shall usually call the subalgebra $\mathcal{T}(V)$ of contravariant tensors the tensor algebra of $V$:

$$\mathcal{T}(V) = \mathbb{R} \oplus V \oplus V \otimes V \oplus V \otimes V \oplus V \otimes V \oplus V \otimes V \oplus V \otimes V \oplus V \otimes V \oplus \ldots$$  \hspace{1cm} (139)

An element of a tensor-product that contains $r$ factors of $V$ and $s$ factors of $V^*$ is also called a tensor of contravariant degree $r$ and covariant degree $s$, or briefly a tensor of type $(r, s)$. The space of all tensors of type $(r, s)$ is denoted $\mathcal{T}^r_s(V)$. [[Note that unless $r = 0$ or $s = 0$ this does not identify the space as e.g. $V \otimes V^* \neq V^* \otimes V$ are both of type $(1, 1)$].]

In addition to linear combinations and tensor products, the tensor algebra is equipped with the following operation which generalizes the trace of a linear map (or of a matrix):
Definition 3.26 Consider a tensor product \( V_1 \otimes \ldots \otimes V_{r+s} \) of type \((r, s)\), i.e. where \( r \) of the \( V_k \) are equal to \( V \) and \( s \) of the \( V_k \) are equal to \( V^* \) and in particular \( V_1 = V \) and \( V_j = V^* \). The contraction (of the \( i \)-th contravariant and \( j \)-th covariant index) of tensors in this space is defined by
\[
(v_1 \otimes \ldots \otimes v_{r+s}) \mapsto \begin{cases} v_j(v_i) \cdot (v_1 \otimes \ldots v_{i-1} \otimes v_{i+1} \ldots v_{j-1} \otimes v_{j+1} \otimes \ldots \otimes v_{r+s}) & \text{if } i < j \\ v_j(v_i) \cdot (v_1 \otimes \ldots v_{j-1} \otimes v_{j+1} \ldots v_{i-1} \otimes v_{i+1} \otimes \ldots \otimes v_{r+s}) & \text{else.} \end{cases}
\] (140)

Proposition 3.33 Contractions are multi-linear maps. A contraction of a tensor of type \((r, s)\) is again a tensor, and it is a tensor of type \((r - 1, s - 1)\).

A very useful result is that every linear map between vector spaces uniquely extends to tensor products. This is an immediate corollary of a fundamental algebraic property:

Proposition 3.34 If \( A \) is an associative algebra with 1 over \( \mathbb{R} \) and \( \phi : V \mapsto \mathbb{R} \) is linear then there exists a unique extension \( \Phi : T(V) \mapsto A \) which is linear, which preserves products, and which is such that \( \Phi(1) = 1 \).

Corollary 3.35 Any \( \phi \in \text{Hom}(V, W) \) extends to a unique homomorphism \( \Phi : T(V) \mapsto T(W) \).

The universal example is the map that maps a basis of an \( n \)-dimensional vector space \( V \) to a set \( x_1, \ldots, x_n \) of noncommuting indeterminates. Its unique extension maps the tensor algebra \( T(V) \) to the algebra of noncommuting polynomials in \( x_1, \ldots, x_n \).

Arguably the most useful subspace of the tensor algebra is the space of alternating or antisymmetric tensors – it provides a very concise and elegant description of higher order differential forms. Along the way it is convenient to also define symmetric tensors which have less dramatic properties.

In the following we shall consider a fixed (finite dimensional) vector space \( V \) (which in differential geometry typically stands for either \( T_p M \) or \( T^*_p M \)). Write \( \Sigma_k \) for the symmetric group, or group of permutations of \( k \) letters. The group \( \Sigma_k \) naturally acts on \( k \)-fold tensor products of \( V \). For \( \sigma \in \Sigma_k \) define a map \( \tilde{\sigma} : T_k(V) \mapsto T_k(V) \) on generators by \([\text{compare polynomials/monomials}]
\[
\tilde{\sigma}(v_1 \otimes \ldots \otimes v_n) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \quad \text{for } v_j \in V
\] (141)

and extend linearly to \( T_k(V) \).

Definition 3.27 A tensor \( x \in T_k(V) \) is called symmetric if \( \tilde{\sigma}(x) = x \) for all \( \sigma \in \Sigma_k \) and it is called alternating (or skew symmetric or antisymmetric) if \( \tilde{\sigma}(x) = \text{sgn}(\sigma)x \) for all \( \sigma \in \Sigma_k \). Write \( S_k(V) \) and \( A_k(V) \) for the subsets of all symmetric and alternating tensors in \( T_k(V) \), respectively. It is convenient to also define the set of alternating \( k \)-forms \( \Lambda^k(V) = A_k(V^*) \subseteq T^k(V) \).

Definition 3.28 A linear map \( L : T_k(V) \mapsto W \) is called symmetric (resp. alternating) if
\[
L \circ \tilde{\sigma} = L \quad (\text{resp. } L \circ \tilde{\sigma} = \text{sgn}(\sigma)L) \quad \text{for all } \sigma \in \Sigma_k
\] (142)

Proposition 3.36 The maps \( \text{Sym}_k, \text{Alt}_k : T_k(V) \mapsto T_k(V) \) defined by
\[
\text{Sym}_k(x) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \tilde{\sigma}(x) \quad \text{and} \quad \text{Alt}_k(x) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \text{sgn}(\sigma) \cdot \tilde{\sigma}(x)
\] (143)

are projections onto the symmetric and alternating tensors \( S_k(V) \) and \( A_k(V) \), respectively.
Exercise 3.51 Verify that \( S_k(V) \) and \( \mathcal{A}_k(V) \) are subspaces of \( T_k(V) \), and prove proposition 3.36.

Proposition 3.37 Suppose \( \Phi : V \rightarrow W \) is a linear map between vector spaces. Then the map \( \Phi^* : T(W) \rightarrow T(V) \) defined on generators by \( \Phi^*(w^1 \otimes \ldots \otimes w^k) = (\Phi^* w^1) \otimes \ldots \otimes (\Phi^* w^k) \) restricts for any \( k \geq 0 \) to a map \( \Phi^* \) mapping \( \Lambda^k(W) \) to \( \Lambda^k(V) \).

It is convenient to define \( \Lambda^0(V) = \mathbb{R} \). Note that \( \Lambda^1(V) = V^* \) (there is only one permutation \( \sigma \) in \( \Sigma_1 \), and clearly \( \tilde{\sigma}(x) = 1 \cdot x = \text{sgn}(\sigma) \cdot x \). Also it is not very hard to see that the dimension of \( \Lambda^n(V) \) is one (if \( \dim(V) = n \)) – and a basis for it is the determinant function suitably interpreted – the usual proof in linear algebra [see e.g. Strang] first show that for any pair of alternating functions on \( \mathbb{R}^n \) one is a multiple of the other . . .

Definition 3.29 Define \( \Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \ldots \) to be the direct sum of all alternating forms of all orders \( k \), i.e. the space of all formal linear combinations of alternating forms. Define the wedge product \( \wedge : \Lambda(V) \rightarrow \Lambda(V) \) on homogeneous elements \( \omega \in \Lambda^k(V) \) and \( \eta \in \Lambda^l(V) \) by

\[
\omega \wedge \eta = \frac{(k+\ell)!}{k! \ell!} \text{Alt}(\omega \otimes \eta) \quad \text{and extend linearly to} \ \Lambda(V)
\] (144)

From the construction it is clear that \( \wedge \) is bilinear and that \( \omega \wedge \omega = (-1)^{k \ell} \omega \wedge \eta \). In particular, if \( k \) is odd, then \( \omega \wedge \omega = 0 \). It takes a little more work to show associativity. Intermediate steps are to first establish the following:

Exercise 3.52 Suppose \( x, y \in T(V) \) and \( \text{Alt}(x) = 0 \). Then \( \text{Alt}(x \otimes y) = \text{Alt}(y \otimes x) = 0 \).

Suppose \( x, y \in T(V) \). Then \( \text{Alt}(\text{Alt}(x \otimes y) \otimes z) = \text{Alt}(x \otimes y \otimes z) = \text{Alt}(x \otimes \text{Alt}(y \otimes z)) \).

Suggestion: Show this for homogeneous tensors \( x \in T^k(V), y \in T^l(V), z \in T^m(V) \) first. Consider the subgroup of permutations that leaves certain elements fixed, and finally use the co-sets of this subgroup.

Proposition 3.38 The vector space \( \Lambda \) with the wedge product \( \wedge \) is an associative algebra.

Theorem 3.39 Suppose \( \{\lambda^1, \ldots, \lambda^n\} \) is a basis for \( V^* \). Then

\[
\{ \lambda^{i_1} \wedge \ldots \wedge \lambda^{i_k} : 1 \leq i_1 \leq \ldots \leq i_k \leq n \}
\] (145)

is a basis for \( \Lambda^k(V) \) and consequently \( \dim \Lambda^k(V) = \binom{n}{k} \).

Proof. Suppose that \( \{v_1, \ldots, v_n\} \) is a basis for the vector space \( V \) and \( \{\lambda^1, \ldots, \lambda^n\} \) is the dual basis for \( V^* \). Suppose \( \lambda \in \Lambda^k \). Then there exist \( c_{i_1 \ldots i_k} \in \mathbb{R} \) such that

\[
\lambda = \text{Alt}(\lambda) = \text{Alt}\left( \sum_{i_1 \ldots i_k \leq n} c_{i_1 \ldots i_k} \lambda^{i_1} \otimes \ldots \otimes \lambda^{i_k} \right) = \sum_{i_1 \ldots i_k \leq n} c_{i_1 \ldots i_k} \text{Alt}\left( \lambda^{i_1} \otimes \ldots \otimes \lambda^{i_k} \right) \] (146)

Each \( \text{Alt}\left( \lambda^{i_1} \otimes \ldots \otimes \lambda^{i_k} \right) \) is either zero or of the form \( \pm \frac{1}{n!} \lambda^{j_1} \wedge \ldots \wedge \lambda^{j_k} \) for some \( 1 \leq j_1 \leq \ldots \leq j_k \leq n \) and hence the elements of this form span \( \Lambda^k(V) \).

On the other hand suppose a linear combination \( \sum_l c_{i_1 \ldots i_k} \lambda^{i_1} \wedge \ldots \wedge \lambda^{i_k} \) is zero, where the sum
extends over all increasing $k$-tupels $I$, i.e. all $(i_1, \ldots, i_k)$ such that $1 \leq i_1 \leq \ldots i_k \leq n$. Then evaluate this linear functional on $v_{j_1} \otimes \ldots \otimes v_{j_k}$ for any $1 \leq j_1, \ldots j_k \leq n$ to obtain

$$0 = \left( \sum_I c_{i_1, \ldots, i_k} \lambda^{i_1} \wedge \ldots \wedge \lambda^{i_k} \right) \left( v_{j_1} \otimes \ldots \otimes v_{j_k} \right)$$

$$= \left( \sum_I c_{i_1, \ldots, i_k} \frac{(1+1)!}{1!1! \ldots 1!} \text{Alt}(\lambda^{i_1} \otimes \ldots \otimes \lambda^{i_k}) \right) \left( v_{j_1} \otimes \ldots \otimes v_{j_k} \right)$$

$$= \left( \sum_i \sum_{\sigma \in S_k} c_{i_1, \ldots, i_k} \text{sgn}(\sigma) \cdot \lambda^{i_1} \otimes \ldots \otimes \lambda^{i_k} \circ \sigma \right) \left( v_{j_1} \otimes \ldots \otimes v_{j_k} \right)$$

$$= c_{j_1, \ldots, j_k}, \quad \text{thus proving linear independence.} \quad (147)$$

**Corollary 3.40** $\lambda^1, \ldots, \lambda^k \in \Lambda^1(V)$ are linearly independent if and only if $\lambda^1 \wedge \ldots \wedge \lambda^k \neq 0$

**Proof.** Suppose that $\lambda^1, \ldots, \lambda^k \in \Lambda^1(V)$ are linearly independent. Then this set may be extended to a basis $\{\lambda^1, \ldots, \lambda^n\}$ of $\Lambda^1(V)$ which is dual to some basis $\{v_1, \ldots v_n\}$ of $V$. Hence

$$\left( \lambda^1 \wedge \ldots \wedge \lambda^k \right) \left( v_1 \otimes \ldots \otimes v_k \right) = \lambda^1(v_1) \ldots \lambda^k(v_k) = 1 \neq 0, \quad (148)$$

i.e. $\lambda^1 \wedge \ldots \wedge \lambda^k$ is a nonzero element of $\Lambda^k(V)$. Conversely, suppose that w.l.o.g. $\lambda^1$ is a linear combination of $\lambda^2, \ldots, \lambda^k$, i.e. there are $c_j$ such that $\lambda^1 = \sum_{j=1}^k c_j \lambda^j$. But then

$$\lambda^1 \wedge \ldots \wedge \lambda^k = \left( \sum_{j=1}^k c_j \lambda^j \right) \wedge \lambda^2 \wedge \ldots \wedge \lambda^k = \sum_{j=1}^k c_j \lambda^j \wedge \left( \lambda^2 \wedge \ldots \wedge \lambda^k \right) = 0. \quad (149)$$
3.15 Exterior derivatives and integrability

This section provides an introduction to the algebra differential $k$-forms on a smooth manifold, and to the exterior derivative $d$. The capstone of this section is an alternative way of characterizing integrability, in terms of differential forms as opposed to tangent vector fields).

The first step is to take the notions of tensor and exterior products, developed in the previous section for a single vector space $V$, to the setting of smooth manifolds. The construction of the associated vector bundles is completely analogous to that of the cotangent bundle: But this time replace the fibres $T_pM$ of the tangent bundle by tensor products $T^k_p(M)$, or by symmetrized tensor products, or alternating products. The following chapter utilizes symmetric tensor fields of type $(0, 2)$, this section is concerned with alternating tensors. We shall slightly abuse notation and write $\Lambda^k(M)$ for the vector bundle whose fibres over $p \in M$ are the spaces $\Lambda^k(T_pM)$, $k \geq 0$ (generally it will be clear from the context what $\Lambda^k$ stands for – with possibly minor confusion arising when the manifold $M$ is a vector space itself …). Moreover it will be convenient to also introduce the vector bundle $\Lambda(M)$ whose fibres are the direct sums

$$\Lambda^0(T_pM) \oplus \Lambda^1(T_pM) \oplus \ldots \oplus \Lambda^m(T_pM)$$  \hspace{1cm} (150)

There are no problems with these bundle constructions, and we proceed to focus our interest on functions that take values in these bundles:

**Definition 3.30** Suppose that $M^m$ is a smooth manifold. A differential $k$-form is a smooth section $\omega: M \mapsto \Lambda^k(M)$. The space of all differential $k$-forms on $M$ is denoted by $\Omega^k(M)$.

This means that $\pi \circ \omega = \text{id}_M$ and for all $X_1, \ldots, X_k \in \Gamma^\infty(M)$ the map $p \mapsto \omega_p(X_{1p}, \ldots, X_{kp})$ is a smooth function on $M$. [[Be warned that the notation and terminology vary substantially among major texts. For example, Spivak uses $\Omega^k$ to denote the space that are $\Lambda^k$ in our notation but does not introduce any special symbol for the space of to differential $k$-forms.]]

Again it is convenient to allow formal sums of differential forms of different degrees and we write

$$\Omega = \Omega^0(M) \oplus \Omega^1(M) \oplus \ldots \oplus \Omega^m(M)$$  \hspace{1cm} (151)

for the space of all such formal sums. One naturally defines sums and exterior products of differential forms pointwise, e.g. $(\omega + \eta)(p) = \omega_p + \eta_p$ and $(\omega \wedge \eta)(p) = \omega_p \wedge \eta_p$.

**Exercise 3.53** Suppose $\Phi \in C^\infty(M, N)$ and $\eta \in \Omega^k(N)$. Verify that $\Phi^* \eta$ defined pointwise by $(\Phi^* \eta)(p) = \Phi^*_p(\eta_{\Phi(p)})$ is a (smooth) differential $k$-form. Compare proposition 3.37 for the definition of $\Phi^*_p(\eta_{\Phi(p)})$, i.e., use $(\Phi^*_p(\eta_{\Phi(p)}))(X_{1p}, \ldots, X_{kp}) = \eta(\Phi_{sp}X_{1p}, \ldots, \Phi_{sp}X_{kp})$.

With these definitions one easily verifies the following:

**Proposition 3.41** Suppose $\omega, \omega' \in \Omega^k(N)$, $\eta \in \Omega^\ell(N)$, $f \in C^\infty(N)$, and $\Phi \in C^\infty(M, N)$. Then

$$\begin{array}{lcl}
(\omega + \omega') \wedge \eta &=& \omega \wedge \eta + \omega' \wedge \eta \\
\omega \wedge \eta &=& (-1)^{k\ell} \cdot \eta \wedge \omega \\
(f \omega) \wedge \eta &=& f(\omega \wedge \eta) = \omega \wedge (f \eta) \\
\Phi^*(\omega \wedge \eta) &=& (\Phi^* \omega) \wedge (\Phi^* \eta)
\end{array}$$  \hspace{1cm} (152)
Recall from 3.10 that locally, at every point $q$ in a chart $(u, U)$, a basis for the cotangent space is given by $\{du_q^1, \ldots, du_q^n\}$. Consequently, the set

$$\{du_q^{i_1} \wedge du_q^{i_2} \wedge \ldots \wedge du_q^{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$$  \hspace{1cm} (153)

spans the vector space $\Lambda^k(T_q M)$ for each $q \in U$. It is convenient to introduce the multi-index $I = (i_1, i_2, \ldots, i_k)$ and write $du^I$ for the differential $k$-form $du^{i_1} \wedge du^{i_2} \wedge \ldots \wedge du^{i_k}$ which is defined for $q \in U$ by $du^I(q) = du^{i_1}_{q} \wedge du^{i_2}_{q} \wedge \ldots \wedge du^{i_k}_{q}$.

Consequently, for every $\omega \in \Omega^k(M)$ there exist (in a chart $(u, U)$ smooth functions $\omega_I \in C^\infty(U)$ such that $\omega = \sum_I \omega_I du^I$ where the sum is again taken over all multi-indices $I = (i_1, i_2, \ldots, i_k)$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

**Exercise 3.54** Suppose that $\omega \in \Omega^k(M)$ and $(u, U), (v, V)$ are charts about $p$. Let $\omega_I, \omega'_I$ be smooth functions on $U \cap V$ such that $\omega = \sum_I \omega_I du^I = \sum_I \omega'_I du^I$. Calculate a formula for $\omega'_I$ in terms of $\omega_I$ and the partial derivatives $\frac{\partial}{\partial u^j}$.

An important special case is when $k = m$ which determines the *volume form* on the manifold:

**Proposition 3.42** Suppose $(u, U)$ and $(v, V)$ are charts for $M^m$. Then

$$dv^1 \wedge dv^2 \wedge \ldots \wedge dv^m = \det\left(\frac{\partial v^j}{\partial u^i}\right)_{i,j=1,\ldots,m} du^1 \wedge du^2 \wedge \ldots \wedge du^m$$  \hspace{1cm} (154)

**Proof.** Suppose $(u, U)$ and $(v, V)$ are charts for $M^m$ and $p \in U \cap V \neq \emptyset$. Use the duality $du^I(\frac{\partial}{\partial u^j}) = \delta^I_j$ and $dv^I(\frac{\partial}{\partial v^j}) = \delta^I_j$, and the transformation formula $\frac{\partial}{\partial v^j} = \sum_{j=1}^m \frac{\partial u^i}{\partial v^j} \frac{\partial}{\partial u^i}$ to calculate

$$du^1 \wedge du^2 \wedge \ldots \wedge du^m(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \ldots, \frac{\partial}{\partial u^m}) =$$

$$= \sum_{i_1,\ldots,i_m=1}^m \frac{\partial u^{i_1}}{\partial u^1} \frac{\partial u^{i_2}}{\partial u^2} \ldots \frac{\partial u^{i_m}}{\partial u^m} dv^1 \wedge dv^2 \wedge \ldots \wedge dv^m(\frac{\partial}{\partial v^{i_1}}, \frac{\partial}{\partial v^{i_2}}, \ldots, \frac{\partial}{\partial v^{i_m}})$$

$$= \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) \frac{\partial v^{\sigma(1)}}{\partial u^1} \frac{\partial v^{\sigma(2)}}{\partial u^2} \ldots \frac{\partial v^{\sigma(m)}}{\partial u^m} dv^1 \wedge dv^2 \wedge \ldots \wedge dv^m(\frac{\partial}{\partial v^{\sigma(1)}}, \frac{\partial}{\partial v^{\sigma(2)}}, \ldots, \frac{\partial}{\partial v^{\sigma(m)}})$$

which is equal to the determinant of the Jacobian matrix of partial derivatives corresponding to the coordinate change. Note that in the key step all summands over multi-indices $I = (i_1, i_2, \ldots, i_m)$ that are not permutations vanish due to the alternating nature of the exterior product. ■

**Exercise 3.55** Consider $M = \mathbb{R}^2 \setminus \{(0,0)\}$ with the standard Cartesian coordinates $(x, y)$ and the polar coordinates $(r, \theta)$ defined on a suitable subset. Express $dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$ via direct calculation in terms of $dr \wedge d\theta$. [[The result is known from calculus – but it is instructive to carry out all steps of the calculation starting with $dx = \cos \theta \, dr - r \sin \theta \, d\theta$ etc.]]

Just as in the case of differential one-forms – sections of the cotangent bundle – differential $k$-forms are predestined to be *pulled back* along a smooth map between manifolds, compare exercise 3.53 and proposition 3.53.
Exercise 3.56 Suppose $\Phi \in C^\infty(M^m, N^n)$, $p \in M$ and $(u,U)$ and $(v,V)$ are charts about $p \in M$ and $\Phi(p) \in N$, respectively. Use the familiar formulas $\Phi^* \left( \frac{\partial}{\partial u^i} \big|_p \right) = \sum_{j=1}^m \frac{\partial (\phi^j \circ \Phi)}{\partial u^i} \big|_p \cdot \frac{\partial}{\partial v^j} \big|_{\Phi(p)}$ and thus $(\Phi^* dv^i)_p = \sum_{j=1}^m \frac{\partial (\phi^j \circ \Phi)}{\partial u^i} \big|_p \cdot du^i_p$ to find formulas for the pullbacks

$$\Phi^*(dv^1 \otimes dv^{i_2} \otimes \ldots \otimes dv^{i_k}) \quad \text{and} \quad \Phi^*(dv^1 \wedge dv^{i_2} \wedge \ldots \wedge dv^n)$$

(155) in terms of $du^{i_1} \otimes du^{i_2} \otimes \ldots \otimes du^{i_k}$, and $du^1 \wedge du^{i_2} \wedge \ldots \wedge du^n$, respectively.

One of the most often used versions of pullbacks are those induced by inclusion maps or imbeddings. Specifically, if $M \subseteq N$ is a submanifold and $\iota: M \hookrightarrow N$ is the inclusion map, then every differential form $\omega \in \Omega(M)$ immediately gives rise to a differential form $\iota^* \omega \in \Omega(M)$. This is so commonly used, and so immediate that in many places one simply uses the same symbol $\omega$ for the pull-back $\iota^* \omega$. For example if $X \in \Gamma^\infty(M)$ is a vector field on $M$ and $\omega \in \Omega(N)$ then one often sees $\omega(X)$ for what should have been written more precisely as $(\iota^* \omega)(X)$.

Exercise 3.57 Consider the inclusion map $\iota: S^2 \hookrightarrow \mathbb{R}^3$ of the (imbedded) 2-sphere into 3 dimensional Euclidean space, equipped with the spherical coordinates $(\theta, \phi)$ and the Cartesian coordinates $(x^1, x^2, x^3)$. Calculate the pullbacks $\iota^* (dx^j) = d(x^j \circ \iota)$ for $j = 1, 2, 3$. [[Note, one often sloppily writes $dx^j$ when really referring to $\iota^* (dx^j)$]] Express $d\theta \wedge d\phi$ in terms of $\iota^* (dx^1 \wedge dx^\ell)$ with $1 \leq j < \ell \leq 3$.

Arguably the most important map associated to differential forms is the exterior derivative which maps differential $k$-forms to differential $(k+1)$-forms. There are many intriguing relations between exterior derivatives of differential forms and Lie brackets of (tangent) vector fields – but the superior algebraic properties of the exterior derivative are hard to overestimate. Loosely following Spivak, we discuss three different ways to introduce exterior derivatives.

The exterior derivative of a differential zero-form, i.e. of a smooth function should just be the usual differential. In particular, in a chart $(u,U)$ we want

$$df = \sum_{j=1}^m \frac{\partial f}{\partial u^j} \cdot du^j$$

(156)

From here it is an obvious road to define for any differential $k$-form $\omega$, in a chart $(u,U)$

$$\text{If} \quad \omega = \sum_I \omega_I \cdot du^I \quad \text{then} \quad d\omega = \sum_I \sum_{j=1}^m \frac{\partial \omega_i}{\partial u^j} \cdot du^j \wedge du^I$$

(157)

(where $I = (i_1, i_2, \ldots, i_k)$ is any multi-index with $1 \leq i_j \leq n$). Such a definition has the obvious disadvantage that it requires an uninviting calculation to check that the definition is independent of the choice of local coordinates.

Exercise 3.58 Suppose $\omega$ is a differential form and $(u,U)$, $(v,V)$ are local coordinate charts. Suppose that $\omega = \sum_I \omega_I \cdot du^I = \sum_J \omega'_J \cdot dv^J$. Verify directly that

$$\sum_I \sum_{j=1}^m \frac{\partial \omega_i}{\partial u^j} \cdot du^j \wedge du^I = \sum_J \sum_{j=1}^m \frac{\partial \omega'_j}{\partial v^j} \cdot dv^j \wedge dv^J$$

(158)
More elegant is an axiomatic characterization of the exterior derivative. The most interesting question is whether there is a compelling very short list of properties that uniquely characterizes the desired exterior derivative. Following Spivak:

**Proposition 3.43** Suppose $M$ is a smooth manifold. Then there is a unique map $d: \Omega^k(M) \to \Omega^{k+1}(M)$ that has the following properties:

(i) If $\omega, \eta \in \Omega(M)$ then $d(\omega + \eta) = d\omega + d\eta$.

(ii) If $\omega \in \Omega^k(M)$, $\eta \in \Omega^1(M)$ then $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-)^k \cdot \omega \wedge (d\eta)$.

(iii) $d^2 = 0$.

(iv) If $(u, U)$ is a chart and $f \in C^\infty(U) = \Omega^0(U)$ then $df = \sum_{j=1}^m \frac{\partial f}{\partial u^j} \cdot du^j$.

**Exercise 3.59** Prove proposition 3.43. [[Suppose there are maps $d, d'$ which both satisfy all properties as above. By the first property (additivity) it suffices to consider terms of the form $\omega_I dx^I$. Then compare $d\omega = d'\omega$, using the second and third properties. Finally show by induction that $d$ and $d'$ agree on all differential $k$-forms.]]

The third alternative is to provide an invariant definition.

**Proposition 3.44** Suppose $\omega \in \Omega^k(M)$. There exists a unique differential $(k+1)$-form denoted $d\omega$ that satisfies for all sequences $X_1, X_2, \ldots, X_{k+1} \in \Gamma^\infty(M)$ of smooth vector fields.

$$d\omega(X_1, X_2, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} X_i (\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) + \sum_{1 \leq i < j \leq k+1} (-)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}).$$

**Exercise 3.60** Prove proposition 3.44 (existence and uniqueness). [[The key if to observe that if $\omega \in \Omega^k(M)$, $X_i \in \Gamma^\infty(M)$, and $f \in C^\infty(M)$, then

$$d\omega(X_1, \ldots, X_{i-1}, fX_i, X_{i+1} \ldots X_k) = f \cdot d\omega(X_1, \ldots, X_{i-1}, X_i, X_{i+1} \ldots X_k)$$

]]

**Definition 3.31** Define $d: \omega(M) \mapsto \Omega(M)$ to be the unique linear map that is characterized in proposition 3.44.

**Exercise 3.61** Suppose $d$ is defined as above. Verify that it satisfies the characterization (157).

**Exercise 3.62** Suppose $d$ is defined as above. Verify that it has the properties itemized in 3.43.

In the special case – the most important one – of $k = 1$ one has:

**Corollary 3.45** Suppose $X, Y \in \Gamma^\infty(M)$ and $\omega \in \Omega^1(M)$. Then

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

**Proposition 3.46** Suppose $\Phi \in C^\infty(M, N)$ and $\omega \in \Omega(N)$. Then $\Phi^*(d\omega) = d(\Phi^*\omega) \in \Omega(M)$.

**Exercise 3.63** Prove proposition 3.46. [[Suggestion: One approach is to work in a chart and consider forms of the form $f du^I$ and proceed by induction on the length of $I$. Alternatively, use the invariant definition 3.31.]]
Finally we return to Frobenius integrability theorem. The invariant definition 3.31 of the exterior derivative $d$, and even better corollary 3.45 demonstrate the relationship between the Lie bracket of vector fields and the exterior derivative of differential forms. To utilize this relation it remains to characterize distributions in terms of differential forms: At every point $p \in M$ a smooth distribution determines a subspace $\Delta(p) \subseteq T_p M$. Instead of characterizing this subspace by a set of tangent vectors that span it, one may describe it by the linear functionals $\lambda: T_p M \mapsto \mathbb{R}$ such that the restriction $\lambda|_{\Delta(p)}$ is identically equal to zero.

On the side note that one may associate to a smooth distribution $\Delta$ a co-distribution $\Delta^\perp$ which assigns to each $p$ the subspace (!) of all $\lambda \in T_p^* M$ that annihilate $\Delta$. One may again define smooth co-distributions as those co-distributions that are spanned by smooth differential one-forms. Note that while the rank of a smooth distribution (i.e. dim $\Delta(p)$) may decrease at points (or more generally on closed subsets), the rank of a distribution that is defined as

$$\Delta(p) = \{ X_p \in T_p M : \omega_p(X_p) = 0 \text{ for all } \omega(p) \in \Delta^*(p) \} \quad (162)$$

can only increase at points if $\Delta^*$ is a smooth co-distribution.

In addition to this linear structure of the set of annihilating differential one-forms, one may also consider the algebra structure of the annihilating differential $k$-forms. The exercise motivates the subsequent definition that associates with $\Delta$ an ideal which provides for particularly elegant statements of integrability conditions – analogous to the concept of involutivity.

**Exercise 3.64** Suppose $\Delta$ is a smooth distribution on $M$, $f \in C^\infty(M)$, and $\omega, \eta \in \Omega^1$ annihilate $\Delta$, i.e. $\omega_p(X_p) = \eta_p(X_p) = 0$ for all $X_p \in \Delta(p)$. Verify that

- $(f \omega + \eta)_p(X_p) = 0$ for all $X_p \in \Delta(p)$, and
- $(\omega \wedge \eta)_p(Y_p, X_p) = 0$, for all $X_p, Y_p \in \Delta(p)$ and all $\theta \in \Omega^1(M)$.

**Definition 3.32** Suppose that $\Delta$ is a distribution on $M$. Let $\mathcal{J}(\Delta) \subseteq \Omega(M)$ be the ideal that is generated by all differential forms $\omega \in \Omega(M)$ such that

$$\omega(X_1, \ldots, X_k) = 0 \text{ whenever } X_1, \ldots, X_k \text{ belong to } \Delta \quad (163)$$

**Theorem 3.47 (Frobenius)** Suppose $\Delta$ is a smooth distribution and $\mathcal{J}(\Delta)$ is the subring (actually, by exercise 3.64 $\mathcal{J}(\Delta)$ is an ideal) of $\Omega(M)$ generated by all smooth differential forms $\omega$ such that $\omega(X_1, \ldots, X_q) = 0$ for all smooth vector fields $X_j \in \Gamma^\infty(M)$ that belong to $\Delta$. Then $\Delta$ is integrable if and only if $d(\mathcal{J}(\Delta)) \subseteq \mathcal{J}(\Delta)$. 

**Corollary 3.48** Suppose $\Delta$ is a smooth distribution on $M$, $p \in M$, and $\omega^{k+1}, \ldots, \omega^m \in \Omega^1$ generate $\mathcal{J}(\Delta)$ on a neighborhood of $p$. Then $\Delta$ is integrable around $p$ if and only if there exist $\theta^j_i \in \Omega^1$ such that

$$d\omega^i = \sum_j \theta^j_i \wedge \omega^j \text{ for all } i = k + 1, \ldots, m. \quad (164)$$

**Proof.** (of Frobenius’ theorem). Clearly $\mathcal{J}(\Delta)$ is locally generated by smooth one-forms. Thus suppose $p \in M$ and $\omega^{k+1}, \ldots, \omega^m \in \Omega^1$ generate $\mathcal{J}(\Delta)$ on some neighborhood $U$ of $p$. Extend to a co-frame $\omega^1, \ldots, \omega^k, \omega^{k+1}, \ldots, \omega^m \in \Omega^1$, i.e. such that $\{\omega^1_q, \ldots, \omega^m_q\}$ is a basis for each $q \in U$. There exist smooth vector fields $X_1, \ldots, X_m \in \Gamma^\infty(M)$ such that $\omega^i(X_j) = \delta^i_j$ on $U$ for all $i, j \leq m$. Clearly the vector fields $X_1, \ldots, X_k$ span $\Delta$ on $U$. 

Recall the earlier version of Frobenius theorem which said that $\Delta$ is integrable if and only if there are functions $c^i_{ij} \in C^\infty(M)$ such that $[X_i, X_j] = \sum_{\ell=1}^m c^i_{ij}X_\ell$. Calculate
\[ d\omega^r(X_i, X_j) = X_i(\omega^r(X_j)) - X_j(\omega^r(X_i)) - \omega^r([X_i, X_j]) \quad (165) \]
Thus if $i, j \leq k$ and $r > k$ then the first two terms on the right hand side vanish. If $\Delta$ is integrable then the last term may be rewritten as
\[ -\sum_{\ell=1}^k c^i_{ij}\omega^r X_\ell = 0 \quad (166) \]
and it is zero since $\omega^r$ annihilates $\Delta$.
Conversely, if $d\omega^r(X_i, X_j) = 0$ for all $r > k$ then the third term on the right hand side must be zero and one concludes that $[X_i, X_j]$ belongs to $\Delta$, i.e. $[X_i, X_j]$ may be rewritten as a (smooth) linear combination of $X_1, \ldots, X_\ell$ and by Frobenius theorem, first version, $\Delta$ is integrable. ■

Frobenius integrability theorem clearly relies in an essential way on the key property (iii) of proposition 3.43 that $d^2 = 0$ which may be considered an elegant invariant way to restate that mixed partial derivatives (of smooth functions) are equal. Since $d\omega = 0$ if $\omega = d\eta$ for some $\eta \in \Omega(M)$, one naturally may ask whether $d\omega = 0$ implies that there exists $\eta \in \Omega(M)$ such that $\omega = d\eta$.

**Definition 3.33** A differential form $\omega \in \Omega(M)$ is called closed if $d\omega = 0$ and is called exact if there exists $\eta \in \Omega$ such that $\omega = d\eta$.

In this language the above may be restated as every exact form is closed and the question is whether a closed form is necessarily exact. In general the answer is negative:

**Exercise 3.65** Consider $\omega = \frac{1}{x^2+y^2}(-y\,dx + x\,dy) \in \Omega^1(M)$ where $M = \mathbb{R}^2 \setminus \{(0,0)\}$. Verify that $d\omega = 0$, and that on any set $U_\alpha = \{ (r\cos \theta, r\sin \theta) : r > 0, \alpha - \pi < \theta < \alpha + \pi \}$ there is a function $\Theta \in \Omega^0(U_\alpha)$ such that $\omega = d\Theta$. Show that, on the other hand, there does not exist any (continuous!) $f \in \Omega^0(M)$ such that $\omega = df$.

As this exercise suggests the question which closed forms are also exact is intimately linked to the topology of the manifold. In the example in the exercise the nonexactness is traced back to the manifold not being simply connected (or not being smoothly contractible to a point. [[The further study of this question leads one to generalize line integrals, Stokes' theorem etc. to differentiable manifolds. Eventually one is led to define an equivalence relation by $\omega \sim \eta$ if $\eta - \omega$ is closed. The deRham cohomology formalizes the algebraic properties of the associated equivalence classes, and connects them to global topological properties such as “the number of holes” in the manifold (using colloquial language.)]]

**Exercise 3.66** Explore the statement that curl (grad $f$) = 0 and div (curl $F$) = 0 are just special cases of the general property $d^2 = 0$. [[For a function $f \in C^\infty(\mathbb{R}^3)$ it is easy to relate $df$ and the gradient of $f$ (are they equal, or is the gradient a tangent vector field?)]. In the case of a vector field $F(x) = (F_1(x), F_2(x), F_3(x))$ discuss different possible points of view, e.g. considering $\vec{F} = F_1dx^1 + F_2dx^2 + F_3dx^3$ and $\vec{F}' = F_1dx^2 \land dx^3 + F_2dx^3 \land dx^1 + F_3dx^1 \land dx^2$. Calculate and discuss $d\vec{F}$ and $d\vec{F}'$. In this context discuss the domain and codomain of such (questionable?) compositions as curl (curl $F$) and $\Delta f = \text{div (grad } f\text{)}$. – Note that a complete discussion of these relationships needs to take into account the Riemannian structure of $\mathbb{R}^3$ and thus is a natural topic of the next chapter. ]]
Application to / interpretation in terms of partial differential equations

volume form, orientation, and integration

in progress