1. If $Y$ is a nonsingular square matrix, then a matrix $X$ is called a logarithm of $Y$ if $\exp(X) = Y$.
   
a. Use the definition of the exponential to show that if $J$ and $Q$ are invertible square matrices of the same size, and $L$ is a logarithm of $J$, then $QLQ^{-1}$ is a logarithm of $QJQ^{-1}$.
   
b. Use the definition of the exponential to show that if $A$ and $B$ are invertible matrices that commute with each other, and $R$ and $S$ are logarithms of $A$ and $B$, then $R + S$ is a logarithm of $AB$.
   
c. Verify that the matrix $Y$ given below has a single eigenvalue $\lambda > 0$. Use the formal power series for the logarithm to find a matrix $X$ such as $\exp(X) = Y$.
   
   Suggestion: Rewrite $Y = \lambda I + N = (\lambda I) \cdot (I + \frac{1}{\lambda}N)$ and use part b. and nilpotency of $N$.
   
   
   $$Y = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
   
   
d. For the matrices $A$ and $B$ given above find a matrix $C$ such that $\exp(C) = \exp(A) \exp(B)$.
   
   Suggestion: Use the formal power series expansion for the logarithm and nilpotency of $A$ and $B$.
   
   e. Look up (e.g. in wikipedia) the BakerCampbellHausdorff (CBH) and Zassenhaus formulas, and find a matrix version of the first. Calculate the commutator $[A, B] = AB - BA$ of the matrices $A$ and $B$ from above, and compare your answer from d. with the expression obtained using the CBH formula.
   
   Comment: In our class we do not have real use of logarithms for matrices other than in Floquet theory. However, the above is intended as a good exercise to practice manipulating exponentials of matrices. The CBH formula is very useful in control theory where one commonly switches between different differential equations. Indeed, these exponentials generalize from matrices to general nonlinear differential equations.
   
2. Review various ways to show positive invariance of a set under the flow of a differential equation $y' = f(y)$ for a smooth vector field $f: \mathbb{R}^2 \rightarrow \mathbb{R}^n$. Your conditions should be given in terms of gradients, derivatives, and inner products and be proven, generally using some form of the chain rule.
   
   a. If $I \subseteq \mathbb{R}$ is an interval and $\phi: I \rightarrow \mathbb{R}^n$, when is the image of $I$ under $\phi$ invariant?
   
   b. If $h \in C^1(\mathbb{R}^n, \mathbb{R})$, $c \in \mathbb{R}$ and $(Dh)$ is nonzero on $M = h^{-1}(c)$, when is $M$ invariant, and when is $\Omega = h^{-1}((\infty, c))$ positively invariant?
   
   c. Explain how stable/unstable manifolds help identify invariant regions. Give a pictorial example.
   
   d. [compare week 1]. Show that for suitable values $a, b \in \mathbb{R}$ the semi-infinite strip $[0, \infty) \times [a, b]$ is positively invariant for the system $x' = 1, y' = y(1 - y) - (1 + \sin 2\pi x)/10$. Does the system have a periodic orbit?
   
   e. [Part of qualifier spring 2003]. Find a nontrivial compact positively invariant set for the system $x' = x + y - x(x^2 + y^2), \quad y' = -x + y - y(x^2 + y^2),$ and find all $\omega$-limit sets of the orbits.
   
   Suggestion: Use polar coordinates and the Poincaré-Bendixon theorem.
   
   f. [Part of qualifier fall 2005]. Show that for $c \geq 0$ sufficiently large the triangle with vertices $(0, 0)$, $(c, 0)$, and $(0, c)$, is positively invariant for the system $x' = \frac{2y}{1+y} - x, \quad y' = \frac{2x}{1+x} - y$.
   
   Find the $\omega$-limit sets of every orbit starting inside this triangle – make sure to check for periodic orbits.
   
3. [Khalil p.309 exercises 7.1a and 7.1 c] Show that each of following systems has a periodic orbit.
   
   a. $x' = y, \quad y' = -x + y(1 - 3x^2 - 2y^2)$.
   
   b. $x' = x + y - x(|x| + |y|), \quad y' = -2x + y - y(|x| + |y|)$.
   
4. [Qualifier fall 2006] Show that the system $x' = \cos x + \cos y - \sin 2x, \quad y' = -\cos x + \cos y - \sin 2y$ has a periodic orbit inside the square $[0, \pi] \times [0, \pi]$. Explicitly verify the positive invariance of the square.

Due Monday Nov 9, 2009.