1. Find lower and upper bounds $0 < T_L < T_u$ for the maximal value $T$ such that the initial value problem $y' = y^2 + \cos y$, $y(0) = 0$ has a solution on the interval $[0, T)$. (Suggestion: Compare with $y' = C \cdot (1 + y^2)$.

2. Discuss the differentiability and Lipschitz continuity properties of the function $f: \mathbb{R} \mapsto \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$. State a (nontrivial) theorem that relates local Lipschitz continuity to (some notion of) differentiability, and prove the theorem.

3. (qual. late 1990s?) Suppose $f: \mathbb{R}^2 \mapsto \mathbb{R}$ is continuously differentiable, $f(0,0) = 0$, and $f$ is globally Lipschitz continuous with Lipschitz constant $L > 0$. Prove that any periodic solution of $(x', y') = f(x, y)$ that encircles $(0, 0)$ must have period no smaller than $\frac{2\pi}{L}$. (Suggestion: Use polar coordinates).

4. (qual. spring 2004) Suppose $f, g: \mathbb{R}^2 \mapsto \mathbb{R}$ are continuous, $f$ is globally Lipschitz continuous in the second argument with Lipschitz constant $L > 0$, and $t_0, x_0, y_0 \in \mathbb{R}$ and $\varepsilon, T > 0$. Suppose that $\phi, \psi: [t_0 + T] \mapsto \mathbb{R}$ are solutions of the pair of initial value problems

$$y' = f(t, y), \quad y(t_0) = x_0,$$
$$y' = g(t, y), \quad y(t_0) = y_0,$$

respectively.

Show that if for all $(t, w) \in [t_0, t_0 + T] \times \mathbb{R}$, $|f(t, w) - g(t, w)| \leq \varepsilon$ then for all $t \in [t_0 + T]

$$|\phi(t) - \psi(t)| \leq (|x_0 - y_0| + \varepsilon T)e^{LT}.$$

5. (qual. Jan. 2002) Explicitly calculate the Picard iterates of the solution of the initial value problem $y' = y$, $y(0) = 1$ starting with $\phi_0(t) = \cos t$ as the initial approximation.

Do the successive approximations converge? If so, what is the limit. Justify your answers.

6. Rewrite the differential equation $z'' + z = 0$ and initial conditions $z(0) = \alpha$, $z'(0) = \beta$ as a system of first order equations and explicitly calculate the Picard iterates, and the limit of the successive approximation process.

7. (qual Aug 1998) Suppose $f: \mathbb{R}^2 \mapsto \mathbb{R}$ is continuous and Lipschitz continuous with Lipschitz constant $L > 0$ in the second argument, and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. Let $\mathcal{F} = C^0([t_0, t_1])$ be the set of real-valued continuous functions of $[t_0, t_1]$ and $\beta > 0$. Prove that if $d: \mathcal{F} \times \mathcal{F} \mapsto \mathbb{R}$ is defined by

$$d(x, y) = \max_{t_0 \leq t \leq t_1} e^{-\beta(t-t_0)}|x(t) - y(t)|$$

then $(\mathcal{F}, d)$ is a metric space. Show that if $\beta > L$ then the map $\Phi: \mathcal{F} \mapsto \mathcal{F}$ defined by

$$\Phi(\phi)(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s)) \, ds$$

is a contraction. Conclude that for every $(t_0, y_0)$ there exists a unique function $\phi: \mathbb{R} \mapsto \mathbb{R}$ satisfying $\phi(t_0) = y_0$ and for all $t \in \mathbb{R}$, $\phi'(t) = f(t, \phi(t))$.

8. Write an outline of the key ideas in a proof of Ascoli’s theorem.